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The many faces of Ocneanu cells

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We define generalised chiral vertex operators covariant under the Ocneanu "double triangle algebra" $\mathcal{A}$, a novel quantum symmetry intrinsic to a given rational 2-d conformal field theory. This provides a chiral approach, which, unlike the conventional one, makes explicit various algebraic structures encountered previously in the study of these theories and of the associated critical lattice models, and thus allows their unified treatment. The triangular Ocneanu cells, the $3j$-symbols of the weak Hopf algebra $\mathcal{A}$, reappear in several guises. With $\mathcal{A}$ and its dual algebra $\hat{\mathcal{A}}$ one associates a pair of graphs, G and $\widetilde{G}$. While G are known to encode complete sets of conformal boundary states, the Ocneanu graphs $\widetilde{G}$ classify twisted torus partition functions. The fusion algebra of the twist operators provides the data determining $\hat{\mathcal{A}}$. The study of bulk field correlators in the presence of twists reveals that the Ocneanu graph quantum symmetry gives also an information on the field operator algebra.

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1. Introduction

This paper stems from the desire to understand Ocneanu recent work on "quantum groupoids" [1,2], also called, in a loose sense, "finite subgroups of the quantum groups", and to reformulate and to exploit it in the context of 2d rational conformal field theories (RCFT). Our approach is inspired by the study of boundary conditions in CFT, either on manifolds with boundaries, or on closed manifolds (e.g. a torus) where the introduction of defect lines (or twists) is possible.

In Boundary CFT (BCFT), the type of boundary states and the corresponding character multiplicities in cylinder partition functions are conveniently encoded in a graph (or a set of graphs) G [3], with vertices denoted by a, b. More precisely the adjacency matrices of the graphs are given by a set of (non negative integer valued) matrices $n_i = \{n_{ia}{}^b\}$ forming a representation of the Verlinde fusion algebra

$$n_i\, n_j = \sum_k N_{ij}{}^k\, n_k\,, \tag{1.1}$$

and it is usually sufficient to specify only a "fundamental" subset of them, which generates the other through fusion.

In accordance with these data we define generalised chiral vertex operators (GCVO), covariant under Ocneanu "double triangle algebra" (DTA) $\mathcal{A}$, a finite dimensional "C^* weak Hopf algebra" (WHA) in the axiomatic setting of [4]. They can be looked at as extensions to the complex plane of the boundary fields and at the same time they yield a precise operator meaning to these fields. The fact that the GCVO have nontrivial braiding allows to give a global operator definition of the half-plane bulk fields, described in the traditional approach only through their small distance (vanishing imaginary coordinate) expansion. The bulk fields are defined as compositions of two generalised or conventional CVO, which makes the construction of their correlators and the derivation of the equations they satisfy straightforward.

The $3j$-symbols ${}^{(1)}F$ of the Ocneanu quantum symmetry, also called "cells", reproduce the boundary field operator product expansion (OPE) coefficients, while the $6j$- symbols F coincide with the fusing matrices, i.e., the OPE coefficients of the conventional CVO. In the "diagonal theories", in which each local field is left-right symmetric, there is a one to one correspondence between the set $\mathcal{I}$ and the spectrum of orthonormal boundary states; then (1.1) is realised by the Verlinde matrices themselves and the two symbols F and ${}^{(1)}F$ coincide. The $3j$-symbols diagonalise the braiding matrices of the generalised CVO (the

R matrix of the quantum symmetry). These new braiding matrices are identified with the Boltzmann weights (in the limit $u \to \pm i\infty$ of their spectral parameter) of the critical $sl(n)$ lattice models which generalise the Pasquier ADE lattice models and their fused versions. Once again the $3j$- symbols provide the basic ingredients of these models. In particular their identification with the Ocneanu intertwining cells gives some new solutions for the boundary field OPE coefficients in the exceptional E_r cases of $\widehat{sl}(2)$ theories; for the A and D-series these constants were computed in [5,6].

Through a discussion parallel to that of boundary states, one may also study the allowed twists (or defect lines) on a torus. The compatibility with conformal invariance and a duality argument similar to Cardy's consistency condition [7] restrict the multiplicities $\tilde{V}_{ii'} = \{\tilde{V}_{ii';\,x}{}^{y}\}$ of occurrence of representations (i,i') in the presence of twists x,y, to be now non negative integer valued matrix representations of the *squared* Verlinde fusion algebra [8]:

$$\tilde{V}_{ii'}\tilde{V}_{jj'} = \sum_{k,k'} N_{ij}{}^{k}\, N_{i'j'}{}^{k'}\, \tilde{V}_{kk'}\,, \qquad \tilde{V}^{1}_{ij^*;1} = Z_{ij}\,, \tag{1.2}$$

where Z_{ij} is the modular invariant matrix. Pairs $\tilde{V}_{i1}\,,\tilde{V}_{1i'}$ of these matrices give rise to another graph $\widetilde{G}$ with vertices x,y [1,2]. Combining the concepts of twists and of boundaries, i.e. inserting twists in the presence of boundaries, leads to yet another set of multiplicities, $\tilde{n}_x = \{\tilde{n}_{ax}{}^{b}\}$, which form a matrix realisation of a new, in general non-commutative, fusion algebra:

$$\tilde{n}_x\tilde{n}_y = \sum_{z} \widetilde{N}_{xy}{}^{z}\tilde{n}_z \;\;, \tag{1.3}$$

$$\widetilde{N}_x\widetilde{N}_y = \sum_{z} \widetilde{N}_{xy}{}^{z}\widetilde{N}_z \,. \tag{1.4}$$

This algebra admits an interpretation as the algebra of the twist operators used in the construction of the partition functions in [8]. It is associated with the Ocneanu graph $\widetilde{G}$ in the sense of the relation

$$\tilde{V}_{ij}\,\widetilde{N}_x = \sum_{z} \tilde{V}_{ij;\,x}{}^{z}\,\widetilde{N}_z\,, \tag{1.5}$$

and we shall also refer to it as the $\widetilde{G}$ *graph algebra*. In the cases described by a block-diagonal modular invariant (a diagonal invariant of an extended theory) it possesses subalgebras interpreted as graph algebras of the chiral graph G, and furthermore a subalgebra

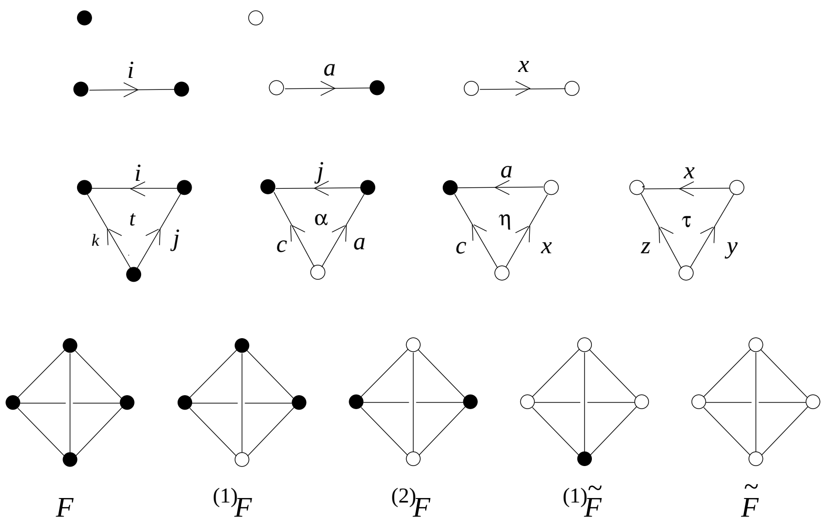

Fig. 1: The simplices

identified with the extended fusion algebra. We find, extending the analysis in [8] to correlators in the presence of twists, that the representations of (1.4) are closely related to the operator product algebra of the physical local fields of arbitrary spin.

In this approach, we see repeated manifestations of the quantum algebra $\mathcal{A}$ and of its dual algebra $\hat{\mathcal{A}}$, both satisfying the axioms of the WHA of [4]. The structure as a whole is maybe most easily described in the combinatorial terms of Ocneanu quantum (co)homology [4] (see also the related notion of "2-category" in [9][1]). The latter considers simplicial 3-complexes built out of the elements depicted on Fig. 1. There are three types of oriented 1-simplices and the triangular 2-simplices come with multiplicities. Each tetrahedral 3-simplex (arrows omitted in Fig. 1) is assigned a $\mathbb{C}$-valued 3-chain, subject to a set of pentagon relations (the "Big Pentagon" of [4]); the middle tetrahedron ${}^{(2)}F$ appears with its inverse, ${}^{(2)}\tilde{F}$, while F , ${}^{(1)}F$, ${}^{(1)}\tilde{F}$, $\tilde{F}$ can be chosen unitary. These data enable one to construct on an abstract level $\mathcal{A}$ and its dual $\hat{\mathcal{A}}$, which are matrix algebras with basis elements represented by two sets of "double triangles", see Fig. 2., related, up to a constant, by ${}^{(2)}F$.

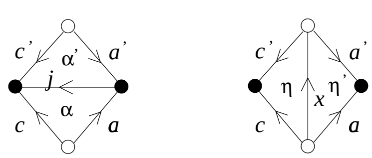

Fig. 2: The double triangles

[1] We thank A. Wassermann for pointing this out to us.

In the present context, the 1-simplices are labelled by the finite set $\mathcal{I}$ of representations of the Verlinde fusion algebra and by the sets $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ of vertices of graphs G and $\widetilde{G}$ of cardinality $|\mathcal{V}| = \mathrm{tr}(Z)$ and $|\widetilde{\mathcal{V}}| = \mathrm{tr}(Z\,{}^t\!Z)$ respectively, where Z is the modular invariant matrix. Each triangular 2-simplex comes with a multiplicity label $t = 1, \cdots, N_{ij}{}^k$, $\alpha = 1, \cdots, n_{ia}{}^c$, $\eta = 1, \cdots, \tilde{n}_{ax}{}^c$, $\tau = 1, \cdots, \widetilde{N}_{xy}{}^z$, and these multiplicities are subject to the relations (1.1) - (1.5). The first two tetrahedra on Fig. 1 represent the $6j$- and the $3j$-symbols F, ${}^{(1)}F$ discussed above.

Thus Ocneanu's double triangle algebra, which is attached specifically to each $2d$ CFT and governs many of its aspects – spectrum multiplicities, structure constants, lattice realisations – appears as its natural quantum symmetry. The problem of identifying the underlying quantum symmetry of a given CFT is by no means new. Several attempts and partial answers were achieved at the end of the 80's and beginning of the 90's, see the discussion below in sections 4 and 5. The previous approaches dealt with the chiral CFT, or equivalently, with the diagonal theories. These are also the only examples of CFT discussed in [4], where the relevance of the WHA as a quantum symmetry was first proposed in the framework of algebraic QFT; in these diagonal cases the four triangle multiplicities above coincide with the Verlinde fusion multiplicities $N_{ij}{}^k$ and accordingly all the tetrahedra on Fig. 1 reduce to the RCFT fusing matrix F. The development of BCFT on one hand side and the work of Ocneanu on the other made available new tools and new ideas; our present considerations yield in particular explicit and non trivial examples of the structure of WHA. The main novelty of the WHA approach is that it has a coassociative coproduct consistent with the CFT fusion rules (the Ocneanu "horizontal" product). The presence of boundaries provides an extension of the Hilbert space of the theory consistent with the fusion rules and basic axioms of the RCFT. At the same time it should be stressed that the parallel with the previous discussions on the "hidden" quantum group symmetry is to some extent superficial, or deceptive, since this is only one of the facets of the Ocneanu symmetry; in contrast to the former the new approach encompasses the full structure of the 2d CFT, so is much richer in content and applications.

We should not conceal, however, that our understanding is still fragmentary. The determination of the cells and of the remaining tetrahedra of Fig. 1 from the complicated set of equations they satisfy poses a difficult technical problem and only partial results in the $\widehat{sl}(2)$ related models are known. Some of the previous quantities, related in particular to the dual structure of the DTA, are still awaiting a better field theoretic interpretation.

Moreover, several of our results are conjectures, tested mainly on the case of $\widehat{sl}(2)$, but lack a general proof. On several of these points, it seems that the approach based on the theory of subfactors [1,10,11] is more systematic. Still, our field theoretic approach provides explicit realisations and exposes some new facts which show the consistency of the whole picture.

This paper is organised as follows: after a brief summary of notations (section 2) we introduce the double triangle algebra (section 3), then define the GCVO and discuss their fusing and braiding properties (section 4). In section 5 we show how the bulk fields may be expressed in terms of GCVO and how the equations they satisfy and the various OPE coefficients may be rederived in a more systematic way. Section 6 discusses briefly the relation to the lattice models and the determination of their Boltzmann weights in terms of the cells. Finally, section 7 deals with the construction of solutions of (1.2)-(1.5) and of the resulting Ocneanu graphs $\widetilde{G}$ and contains a derivation of a formula relating the OPE coefficients of arbitrary spin fields to data of the graph. Details are relegated to two appendices. Sections 5, 6 and 7 may be read independently of one another.

Preliminary accounts of this work have been reported at several conferences (ICMP, London, July 2000; 24th Johns Hopkins Workshop, Budapest, August 2000; TMR Network Conference, Paris, September 2000 [12]; Kyoto Workshop on Modular Invariance, ADE, Subfactors and Geometry of Moduli Spaces, November, 2000) or have been published separately [8]. It should be stressed that this work was strongly influenced by Ocneanu's (unfortunately unpublished) work and that many of the concepts and results presented here originate in his work.

2. Notations

A rational conformal field theory is conventionally described by data of different nature :

• *Chiral data* specify the chiral algebra $\mathfrak{A}$ and its finite set $\mathcal{I}$ of irreducible representations $\mathcal{V}_i$, $i \in \mathcal{I}$, the characters $\chi_i(q) = \mathrm{tr}_{\mathcal{V}_i} q^{L_0 - c/24}$, the unitary and symmetric matrix S_{ij} of modular transformations of the χ, the fusion coefficients ${N_{ij}}^k$, $i, j, k \in \mathcal{I}$, assumed to be given by Verlinde formula

$$ {N_{ij}}^k = \sum_{\ell \in \mathcal{I}} \frac{S_{i\ell} S_{j\ell} S_{k\ell}}{S_{1\ell}} \ . \tag{2.1} $$

Our convention is that the label $i = 1$ refers to the "vacuum representation", and $\mathcal{V}_{i^*}$ denotes the representation conjugate to $\mathcal{V}_i$. Chiral vertex operators $\phi^t_{ij}(z)$ and their fusion and braiding matrices $F_{pt}[{}^{i\,j}_{k\,l}]$ and $B_{pt}[{}^{i\,l}_{k\,j}]$ are also part of the set of chiral data.

• *Spectral data* specify which representations of $\mathfrak{A} \otimes \mathfrak{A}$ appear in the bulk : these data are usually conveniently encoded in the partition function on a torus, with the property of modular invariance

$$Z = \sum_{i,j \in \mathcal{I}} Z_{ij} \chi_i(q) (\chi_j(q))^* \; ; \tag{2.2}$$

here, the integer Z_{ij} specifies the multiplicity of occurrence of $\mathcal{V}_i \otimes \overline{\mathcal{V}_j}$ in the Hilbert space of the theory; unicity of the vacuum is expressed by $Z_{11} = 1$.

• Finally these spectral data must be supplemented by data on the structure constants of the Operator Product Algebra (OPA). This last set of data is the one which is most difficult to determine as it results from the solution of a large system of non linear equations involving the braiding matrices whose general form is in general unknown.

It has been recognized some time ago that these spectral and OPA data have to do with graphs. The latter (ADE Dynkin diagrams and their generalizations) (i) encode in the spectrum of their adjacency matrices the spectral data [13,14,15]; (ii) contain, through the so-called Pasquier algebra, information on the OPA structure constants, see [16,17,18] and below, section 7. In fact these graphs are nothing else than the graphs of adjacency matrices n_i of (1.1). These matrices n_i are diagonalisable in a common orthonormal basis:

$$n_{ia}{}^b = \sum_{j \in \mathrm{Exp}} \frac{S_{ij}}{S_{1j}} \, \psi^j_a \, \psi^{j*}_b \tag{2.3}$$

and obey the identities

$$n_{ia}{}^b = n_{i^* a^*}{}^{b^*} = n_{i^* b}{}^a \,. \tag{2.4}$$

Here and throughout this paper, we make use of the notation Exp to denote the terms appearing in the diagonal part of the modular invariant (2.2)

$$\mathrm{Exp} = \{(j, \alpha), \; \alpha = 1, \cdots, Z_{jj}\} \; . \tag{2.5}$$

The two notations $\psi^{(j,\alpha)}$ and ψ^j, $j \in \mathrm{Exp}$ will be used interchangeably. In the following, ψ^1 refers to the Perron-Frobenius eigenvector, whose components are all positive. Finally, in (2.4), the conjugation of vertices $a \to a^*$ is defined through $\psi^j_{a^*} = (\psi^j_a)^* = \psi^{j^*}_a$.

A particular set of matrices n is provided by the Verlinde matrices N themselves, which form the regular representation of the fusion algebra. This is the diagonal case for which $\mathrm{Exp} = \mathcal{I}$ and the corresponding torus partition function is simply given by $Z_{ij} = \delta_{ij}$.

3. Ocneanu graph quantum algebra

Given a solution of the equation (1.1) consider an auxiliary Hilbert space $V^j \cong \mathbb{C}^{m_j}$ with basis states $|e^{j,\gamma}_{ba}\rangle$, $\gamma = 1, 2, \ldots, n_{ja}{}^b$. It has dimension $m_j = \sum_{a,b} n_{ja}{}^b = \sum_{a,b,\gamma} 1$, in particular $\dim V^1 = \mathrm{tr}(n_1) = |\mathcal{V}|$. A scalar product in $\oplus_{j\in\mathcal{I}} V^j$ is defined as

$$\langle e^{j,\gamma}_{ba} | e^{j',\gamma'}_{b'a'}\rangle = \delta_{bb'}\delta_{aa'}\,\delta_{jj'}\,\delta_{\gamma'\gamma}\sqrt{\frac{P_a\,P_b}{d_j}}\,, \qquad d_j = \frac{S_{j1}}{S_{11}}\,, \quad P_a = \frac{\psi^1_a}{\psi^1_1}\,. \tag{3.1}$$

We define the tensor product decomposition of states $|e^{i,\alpha}_{cb}\rangle \otimes |e^{j,\gamma}_{b'a}\rangle$ for coinciding $b' = b$ according to

$$|e^{i,\alpha}_{cb}\rangle \otimes_h |e^{j,\gamma}_{ba}\rangle = \sum_{k\in\mathcal{I}} \sum_{\beta=1}^{n_{ka}{}^c} \sum_{t=1}^{N_{ij}{}^k} {}^{(1)}F_{bk}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta\, t}_{\alpha\,\gamma} \sqrt{P_b}\,\left(\frac{d_k}{d_i d_j}\right)^{\frac14} |e^{k\,,\beta}_{ca}(ij;t)\rangle\,. \tag{3.2}$$

This is a "truncated" tensor product, in the sense that we restrict to a subspace $V^i \otimes_h V^j$ of $V^i \otimes V^j$, $(cb)\otimes_h (b'a) = \delta_{bb'}\ (cb)\otimes(b'a)$, with $\dim(V^i \otimes_h V^j) = \sum_{a,c} (n_i n_j)_a{}^c \le m_i m_j$. The multiplicity of V^k in $V^i \otimes_h V^j$ is identified with the Verlinde multiplicity $N_{ij}{}^k$. Then the counting of states in both sides of (3.2) is consistent, taking into account (1.1). In (3.2) $e^{k\,,\beta}_{ca}(ij;t)$ give a basis, normalised as in (3.1), for the space V^k in

$$V^i \otimes_h V^j \cong \oplus_k N_{ij}{}^k\, V^k\ . \tag{3.3}$$

The ${}^{(1)}F \in \mathbb{C}$ are Clebsch-Gordan coefficients ("$3j$- symbols"), assumed to satisfy the conditions:

– if one of the indices i or j is equal to 1, the tensor product must trivialise and accordingly

$$ {}^{(1)}F_{bk}\begin{bmatrix} 1 & j \\ c & a \end{bmatrix}^{\beta\, t}_{\alpha\,\gamma} = \delta_{kj}\,\delta_{bc}\,\delta_{\beta\gamma}\delta_{t1}\delta_{\alpha 1}\,; \tag{3.4}$$

– the unitarity conditions, expressing the completeness and orthogonality of the bases in $V^i \otimes_h V^j$

$$\begin{aligned}
&\sum_{b\,,\alpha\,,\gamma} {}^{(1)}F^*_{bk'}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta'\ t'}_{\alpha\,\gamma} {}^{(1)}F_{bk}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta\ t}_{\alpha\,\gamma} = \delta_{kk'}\,\delta_{\beta\,\beta'}\,\delta_{tt'}\,, \\
&\sum_{k\,,\beta\,,t} {}^{(1)}F^*_{b'k}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta\ t}_{\alpha'\,\gamma'} {}^{(1)}F_{bk}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta\ t}_{\alpha\,\gamma} = \delta_{bb'}\,\delta_{\gamma\,\gamma'}\,\delta_{\alpha\,\alpha'}\,,
\end{aligned} \tag{3.5}$$

where ${}^{(1)}F^*$ is the complex conjugate of ${}^{(1)}F$.

In the original (combinatorial) realisation of [1] (for the ADE graphs of the case $sl(2)$), V^j is the linear space of "essential paths of length j" on the graph G. Then (3.2) is interpreted as a composition of essential paths, which is not an essential path in general, but is a linear combination of such paths.

The requirement of associativity of the product (3.2) leads to the "mixed" pentagon relation

$$F\ {}^{(1)}F\ {}^{(1)}F\ = {}^{(1)}F\ {}^{(1)}F\,, \tag{3.6}$$

or, more explicitly

$$\begin{aligned}\sum_{m,\,\beta_2,t_3,t_2} F_{mp}\begin{bmatrix} i & j \\ l & k\end{bmatrix}^{u_2\,u_3}_{t_2\,t_3}\ {}^{(1)}F_{bl}\begin{bmatrix} i & m \\ a & d\end{bmatrix}^{\gamma_1\,t_2}_{\alpha_1\,\beta_2}\ {}^{(1)}F_{cm}\begin{bmatrix} j & k \\ b & d\end{bmatrix}^{\beta_2\,t_3}_{\alpha_2\,\alpha_3} \\ = \sum_{\beta_1} {}^{(1)}F_{cl}\begin{bmatrix} p & k \\ a & d\end{bmatrix}^{\gamma_1\,u_2}_{\beta_1\,\alpha_3}\ {}^{(1)}F_{bp}\begin{bmatrix} i & j \\ a & c\end{bmatrix}^{\beta_1\,u_3}_{\alpha_1\,\alpha_2}\,.\end{aligned} \tag{3.7}$$

Here F is the matrix (the "$6j$-symbols"), unitary in the sense of the analogue of (3.5), relating the two bases in $V^i \otimes_h V^j \otimes_h V^k$. To make contact with the standard notation (cf. e.g. [19]),

$$F^*_{mp}\begin{bmatrix} i & j \\ l & k\end{bmatrix} = \begin{Bmatrix} i & j & p \\ k & l & m\end{Bmatrix}\,. \tag{3.8}$$

There is a gauge freedom in ${}^{(1)}F$ due to the arbitrariness in the choice of basis, $e^{j,\alpha}_{cb} \to \sum_\alpha U^{j,\alpha,\alpha'}_{cb}\, e^{j,\alpha}_{cb}$, where U is an arbitrary unitary matrix. It is useful to have a graphical notation for the $3j$-symbols ${}^{(1)}F$ by means of triangles, and for the $6j$-symbols F by means of tetrahedra (see Fig. 3). Then relations (3.4)-(3.7) are simply depicted [2]. In this graphical representation, the gauge freedom consists in changing any edge $b \xrightarrow{j,\alpha} c$ by a unitary matrix $U^{j,\alpha,\alpha'}_{cb}$.

The pentagon equation (3.7) can be solved for F given the $3j$-symbols ${}^{(1)}F$ and using the unitarity relation (3.5). Conversely, the relation (3.7) can be interpreted, given F, as

[2] The reader should not be confused by the multiplicity of graphical representations used in this paper for the same objects. It turns out that depending on the question, a different representation may be clearer or more profitable. The triangles used here for the "cells" may be regarded as obtained from the tetrahedra of Fig. 1 with three $\bullet$ and one $\circ$ by projecting the three edges $\circ\!\!-\!\!\!-\!\!\bullet$ with their label a, b, c on the triangle with black vertices. Likewise, in the representation of Fig. 1, the pentagon identity (3.7) is depicted by the two ways of cutting a double tetrahedron into two or three tetrahedra.

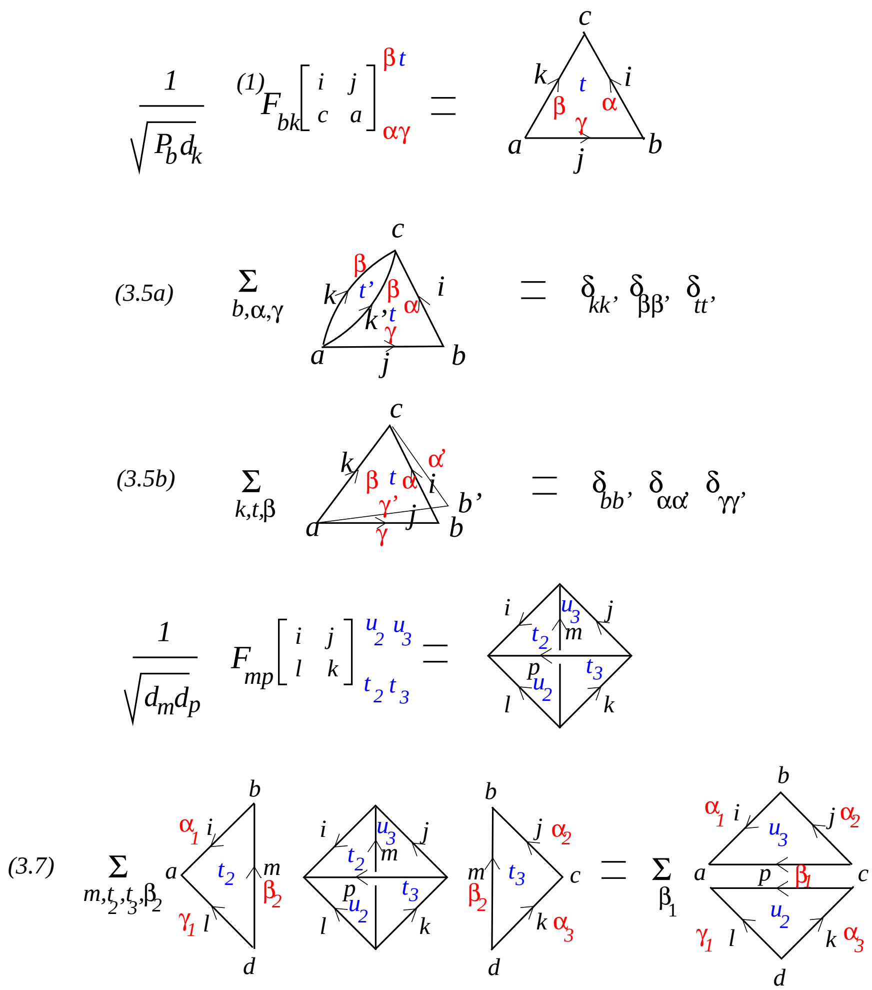

Fig. 3: Graphical representation of the ${}^{(1)}F$ $3j$-symbols, of their orthogonality relations, of the $6j$-symbols and of the pentagon identity (3.7). Factors depending on P_a and d_j have been omitted.

a (recursive) relation for ${}^{(1)}F$. In fact, the matrix F is taken to be the matrix ${}^{(1)}F$ of the diagonal case ($n_i \equiv N_i$), as we identify in that case the $3j$- and the $6j$-symbols and the equation (3.7) coincides then with the standard pentagon identity for F

$$F\ F\ F = F\ F\,. \tag{3.9}$$

We can also look at (3.7) and its solutions ${}^{(1)}F$ as providing more general realisations of the pentagon identity (3.9), corresponding to the matrix representations n_i of the Verlinde fusion algebra (1.1). If we consider along with the states $|e_{ca}^{j,\beta}\rangle$ (triangles with one white and two black vertices), the vector spaces of "diagonal" states $|e_{kj}^{i,t}\rangle$, $t = 1, 2, \ldots N_{ij}{}^k$ (the triangles with three black vertices in Fig. 1), we can identify the basis states in the r.h.s. of (3.2) with the "mixed" products $|e_{ca}^{k,\beta}\rangle \otimes\ |e_{kj}^{i,t}\rangle$.

A solution of (3.9) is determined by the chiral data characterising the CFT. For instance, in the theories based on $\widehat{sl}(2)$, the solution provides the fusing matrices of the CVO and is known to be given in terms of the $6j$ -symbols of the quantum algebra $U_q(sl(2))$, restricted to matrix elements consistent with the fusion rules. For given F the solution of (3.7) is by definition restricted by the data in (1.1), (2.3).

In agreement with the symmetry (2.4) we introduce two (commuting) antilinear involutive maps $V^j \to V^{j^*}$, $(e^{j,\beta}_{ca})^* = e^{j^*,\beta^*}_{c^*a^*}$, and $(e^{j,\beta}_{ca})^+ = e^{j^*,\beta^+}_{ac}$. Correspondingly there are two bilinear forms on $V^{j^*} \otimes V^j$ determined by the sesquilinear form (3.1), i.e., two dual bases in V^{j^*}. The first, given by $*$, corresponds to the complex conjugation of the components of the initial basis in V^j when it is realised through unit vectors in $\mathbb{C}^{m_j}$. The second basis is determined requiring that

$$\sqrt{\frac{d_j}{P_c}}\ \langle e^1_{aa}(j^*j)|e^{j^*,\beta^+}_{ac} \otimes_h e^{j,\beta'}_{ca'}\rangle = \langle\, e^{j,\beta}_{ca}|\, e^{j,\beta'}_{ca'}\,\rangle\,, \tag{3.10}$$

which implies

$${}^{(1)}F_{c1}\begin{bmatrix} j^* & j \\ a & a' \end{bmatrix}^{1\,1}_{\beta^+\,\beta'} = \delta_{aa'}\,\delta_{\beta'\beta}\sqrt{\frac{P_c}{P_a d_j}}\,. \tag{3.11}$$

This is a gauge fixing choice consistent with the unitarity condition (3.5) and the relation

$$d_p\,P_a = \sum_c n_{pa}{}^c\,P_c\,, \tag{3.12}$$

derived from (2.3). In the diagonal case it coincides with the standard gauge fixing of the fusing matrices F of the conformal models based on $\widehat{sl}(n)$. Assuming that on tensor products $(x\otimes y)^* = x^*\otimes y^*$, $(x\otimes y)^+ = y^+\otimes x^+$, and denoting the dual basis states in the tensor product $e^{k^*,\beta^*}_{c^*a^*}(i^*j^*;t^*) := (e^{k,\beta}_{ca}(ij);t)^*$ and $e^{k^*,\beta^+}_{ac}(j^*i^*;\sigma(t^*)) := (e^{k,\beta}_{ca}(ij);t)^+$ (since $N_{j^*i^*}^{k^*} = N_{ij}^k$), these maps imply the symmetry relations for the $3j$-symbols ${}^{(1)}F$

$${}^{(1)}F^*_{bk}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta\ t}_{\alpha\,\gamma} = {}^{(1)}F_{b^*k^*}\begin{bmatrix} i^* & j^* \\ c^* & a^* \end{bmatrix}^{\beta^*\ t^*}_{\alpha^*\,\gamma^*} = {}^{(1)}F_{bk^*}\begin{bmatrix} j^* & i^* \\ a & c \end{bmatrix}^{\beta^+\ \sigma(t)}_{\gamma^+\,\alpha^+}\,, \tag{3.13}$$

while from the pentagon relation taken at $l=1$ and (3.11) one derives

$${}^{(1)}F_{bk}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta\ t}_{\alpha\,\gamma} = \sqrt{\frac{P_b\,d_k}{P_c\,d_j}}\ {}^{(1)}F^*_{cj}\begin{bmatrix} i^* & k \\ b & a \end{bmatrix}^{\gamma\ t^+}_{\alpha^+\,\beta}\,. \tag{3.14}$$

The space $\oplus_{j\in\mathcal{I}}\ \mathrm{End}(V^j)$ is a matrix algebra $\mathcal{A} = \oplus_{j\in\mathcal{I}}\, M_{m_j}$ on which a second product (or a coproduct) is defined via the $3j$-symbols ${}^{(1)}F$ in (3.2). This is the Ocneanu

double triangle algebra [1], an example (and presumably a prototype) of the notion of weak C^* Hopf algebra introduced in [4]; this structure has also received the name of "quantum groupoid" [1,20]; see also [10,11] for recent developments of the original Ocneanu approach. Together with its dual algebra, $\mathcal{A}$ is interpreted in the present context as the quantum symmetry of the CFT, either diagonal or non-diagonal. We review below briefly some basic properties of $\mathcal{A}$ and give further details in appendix A.

The matrix units in M_{m_j} (block matrices in $\mathcal{A}$) are identified with states in $V^j \otimes V^{j^*}$,

$$e_{j;\beta,\beta'}^{(ca)\,,(c'a')} = \frac{\sqrt{d_j}}{(P_c P_a P_{c'} P_{a'})^{\frac{1}{4}}}\ |e_{ca}^{j\,,\beta}\rangle\langle e_{c'a'}^{j\,,\beta'}|\ , \tag{3.15}$$

so that

$$e_{k;\beta',\beta''}^{(c'a')(c''a'')}\ |e_{ca}^{i\,,\beta}\rangle = \delta_{ik}\,\delta_{aa''}\,\delta_{cc''}\,\delta_{\beta\beta''}\,\left(\frac{P_a\,P_c}{P_{a'}\,P_{c'}}\right)^{\frac{1}{4}}\ |e_{c'a'}^{k\,,\beta'}\rangle\ . \tag{3.16}$$

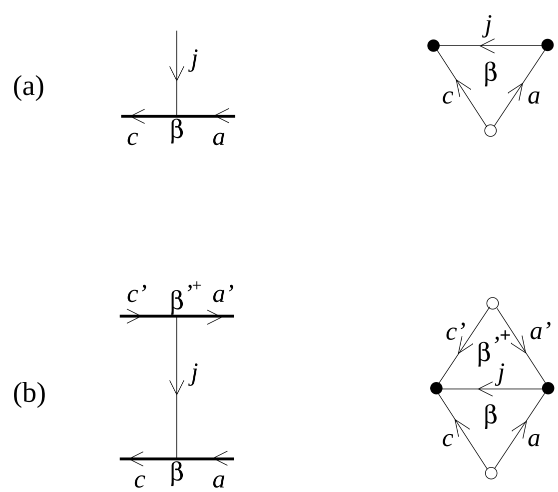

Fig. 4: Two alternative representations of (a) the basis vectors $(\frac{d_j}{P_a\,P_c})^{\frac{1}{4}}\,|e_{ca}^{j,\beta}\rangle$, (b) the matrix units $e_{j;\beta,\beta'}^{(ca),(c'a')}$.

They are depicted as 4-point blocks in Fig. 4, where the states in V^j correspond to 3-point vertices, or, dually, to triangles, whence the name "double triangle algebra" for the algebra $\mathcal{A}$ spanned by the elements (3.15), $j \in \mathcal{I}$. Their matrix ("vertical") multiplication is simply

$$e_{j\,,\beta,\beta'}^{(ca)(c'a')}\ e_{i\,,\gamma',\gamma}^{(d'b')(db)} = \delta_{ij}\,\delta_{a'b'}\,\delta_{c'd'}\,\delta_{\beta'\gamma'}\,e_{j\,,\beta,\gamma}^{(ca)(db)}\ . \tag{3.17}$$

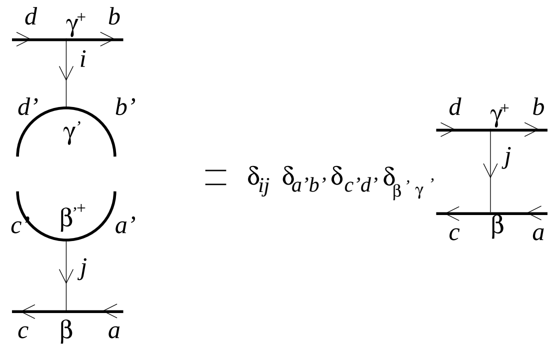

Fig. 5: The vertical product

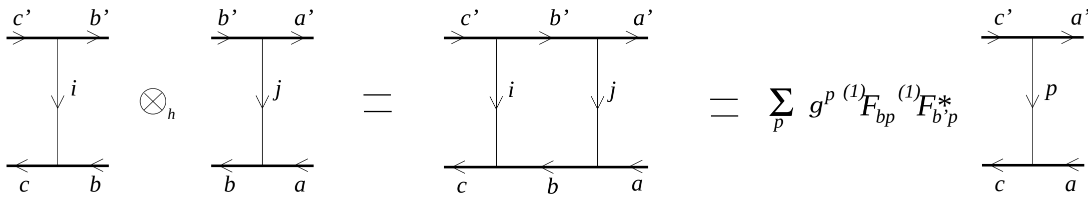

Fig. 6: The horizontal product

The product (3.17) is illustrated on Fig. 5 by composing vertically the blocks representing the two elements (the second above the first), and a similar picture represents (3.16). The identity element 1_v in $\mathcal{A}$ with respect to this multiplication is given by $1_v = \sum_{i\,,c\,,b\,,\alpha} e^{(cb)(cb)}_{i\,,\alpha\,,\alpha}$.

A second, "horizontal", product is defined [1], composing two blocks horizontally, see Fig. 6. Its decomposition is inherited from the r.h.s. of the product $\otimes_h$ in (3.2), and thus the r.h.s. in Fig. 6 involves the $3j$-symbols ${}^{(1)}F$ and ${}^{(1)}F^*$. The normalisation constant is chosen for later convenience as given by $g^{p;b,b'}_{ij} = c_i^{bc'}\, c_j^{ab'}/c_p^{ac'}$, with $c_j^{ab'} = \frac{d_j}{\sqrt{P_a\, P_{b'}}}\, \frac{S_{11}}{\psi_1^1}$. Alternatively we can define [4] a coproduct $\triangle : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$\triangle(e^{(ca)(c'a')}_{k\,,\beta,\beta'}) :=$$
$$\sum_{\substack{i\,,j\\ t}} \; \sum_{\substack{b\,,b'\\ \alpha\,,\alpha'\,,\gamma\,,\gamma'}} {}^{(1)}F^*_{bk} \begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta\, t}_{\alpha\,\gamma} \; {}^{(1)}F_{b'k} \begin{bmatrix} i & j \\ c' & a' \end{bmatrix}^{\beta'\, t}_{\alpha'\,\gamma'} \; e^{(cb)(c'b')}_{i\,,\alpha\,,\alpha'} \otimes e^{(ba)(b'a')}_{j\,,\gamma\,,\gamma'} \,. \tag{3.18}$$

The unitarity (3.5) of ${}^{(1)}F$ ensures that $\triangle(ab) = \triangle(a)\,\triangle(b)$ while the coproduct is coassociative, $(\triangle \otimes Id) \circ \triangle = (Id \otimes \triangle) \circ \triangle$, whenever there exist a unitary F (in the sense of the diagonal analogue of (3.5)), satisfying along with ${}^{(1)}F$ the pentagon identity (3.7).

The “star” operation in $\mathcal{A}$, $(xy)^+ = x^+ y^+$, is inherited from the map $(+)$ defined above,

$$\left(e^{(cb)(c'b')}_{i\,,\alpha,\alpha'}\right)^+ = e^{(bc)(b'c')}_{i^*\,,\alpha^+,\alpha'^+}\,. \tag{3.19}$$

It is a homomorphism of the algebra, i.e. of the vertical product, and an anti-homomorphism of the horizontal product, $(a \otimes_h b)^+ = b^+ \otimes_h a^+$, as well as $(a \otimes b)^+ = b^+ \otimes a^+$, so that $\triangle(a^+) = \triangle(a)^+$. The algebra $\mathcal{A}$ is given a coalgebra structure defining a counit $\varepsilon : \mathcal{A} \to \mathbb{C}$ according to

$$\varepsilon(e^{(ca)(c'a')}_{j\,,\beta,\beta'}) := \delta_{j1}\,\delta_{ac}\,\delta_{a'c'}\,\delta_{\beta 1}\,\delta_{\beta' 1}\,, \tag{3.20}$$

which satisfies the compatibility condition $(\varepsilon \otimes Id)\,\circ\,\triangle = Id = (Id \otimes \varepsilon)\,\circ\,\triangle\,.$

The definitions (3.18), (3.20) imply, however, that $\triangle(1_v) \neq 1_v \otimes 1_v$ and that the counit is *not* a homomorphism of the algebra, $\varepsilon(u)\,\varepsilon(w) \neq \varepsilon(u\,w)$ for general elements $u, w \in \mathcal{A}$, i.e., the DTA is not a Hopf algebra, see the appendix for more details on its structure of “weak Hopf algebra”; in particular the antipode is defined according to

$$S(e^{(cb)(c'b')}_{i\,,\alpha,\alpha'}) = \sqrt{\frac{P_b P_{c'}}{P_{b'} P_c}}\; e^{(b'c')(bc)}_{i^*\,,\alpha'^+,\alpha^+} = \sqrt{\frac{P_b P_{c'}}{P_{b'} P_c}}\; (e^{(c'b')(cb)}_{i\,,\alpha',\alpha})^+\,. \tag{3.21}$$

Using the unitarity (3.5) of the $3j$-symbols ${}^{(1)}F$, it is straightforward to show that the elements

$$\hat{e}_i = \sum_{c\,,b\,,\alpha} \frac{1}{c_i^{bc}}\; e^{(cb)(cb)}_{i\,,\alpha\,,\alpha}\,, \tag{3.22}$$

realise the Verlinde algebra with respect to the horizontal product in $\mathcal{A}$,

$$\hat{e}_i \otimes_h \hat{e}_j = \sum_k N_{ij}{}^k\, \hat{e}_k\,, \tag{3.23}$$

and $\hat{e}_1$ is the identity matrix of $\mathcal{A}$ for that product.

4. Generalised chiral vertex operators

We now return to the field theory. Let $i, j, k \in \mathcal{I}$ s.t. $N_{ij}{}^k \neq 0$ and let и label descendent states in $\mathcal{V}_i$. The chiral vertex operator $\phi^k_{ij,t;\text{и}}(z)$, with t – a basis label, $t = 1, 2, \ldots, N_{ij}{}^k$, is an intertwining operator $\mathcal{V}_j \to \mathcal{V}_k$ [21]. We tensor this field with an intertwining operator $V^j \to V^k$

$$P^{k,\alpha;j,\gamma}_{cb,ab} = \sqrt{\frac{d_j}{P_a\,P_b}}\; |e^{k,\alpha}_{cb}\rangle\langle\, e^{j,\gamma}_{ab}|\,, \tag{4.1}$$

which corresponds to a state in $V^k \otimes_h V^{j^*}$. This defines a generalised chiral vertex operator (GCVO)

$$\oplus_{j\in\mathcal{I}} \mathcal{V}_j \otimes V^j \to \oplus_{k\in\mathcal{I}} \mathcal{V}_k \otimes V^k \,,$$

$$ {}^c\Psi^a_{i,\beta;\text{и}}(z) = \sum_{j\,,k\,,t} \phi^k_{ij,t;\text{и}}(z) \otimes \sum_{b\,,\alpha\,,\gamma} {}^{(1)}F_{ak} \begin{bmatrix} i & j \\ c & b \end{bmatrix}^{\alpha\, t}_{\beta\,\gamma} P^{k,\alpha;j,\gamma}_{cb,ab} \,. \tag{4.2}$$

The projectors (4.1) satisfy

$$\begin{aligned} P^{i,\alpha;k,\gamma}_{cb,ab}\; P^{k',\gamma';j,\delta}_{a'b',db'} &= \delta_{bb'}\,\delta_{aa'}\,\delta_{kk'}\,\delta_{\gamma\gamma'}\,P^{i,\alpha;j,\delta}_{cb,db}\,, \\ P^{k,\alpha;j,\gamma}_{cb,ab}\;|e^{j',\gamma'}_{a'b'}\rangle &= \delta_{bb'}\,\delta_{aa'}\,\delta_{jj'}\,\delta_{\gamma\gamma'}\,|e^{k,\alpha}_{cb}\rangle\,, \\ \langle e^1_{dd}|P^{k,\alpha;j,\gamma}_{cb,ab}|e^1_{d'd'}\rangle &= \delta_{k1}\,\delta_{j1}\,\delta_{cd}\,\delta_{bd}\,\delta_{bd'}\,\delta_{ad'}\,P_a\,. \end{aligned} \tag{4.3}$$

From (4.2), (4.3) we have in particular

$$ {}^c\Psi^a_{j,\beta}(0)\,|0\rangle\otimes|e^1_{aa}\rangle = \phi^j_{j1}(0)|0\rangle\otimes|e^{j,\beta}_{ca}\rangle =: |j\,,\beta\rangle\,, \quad \beta = 1,2\ldots,n_{ja}{}^c \tag{4.4}$$

where $|j\,,\beta\rangle$ is the explicit form of the highest weight state of the chiral algebra representation $\mathcal{V}_{j,\,\beta}$, "augmented" with the additional coupling label β, used in the computation of the cylinder partition function in the Hilbert space $\mathcal{H}_{a|c} = \oplus_{j,\beta}\,\mathcal{V}_{j,\,\beta}$, [3]. The correlators of the generalised CVO (4.2) are computed projecting on "vacuum" states $|0\rangle\otimes|e^1_{aa}\rangle$ in the space $\mathcal{V}_1\otimes V^1$; recall that V^1 has a nontrivial dimension $|\mathcal{V}|$. Since $P^{1,1;1,1}_{ab,db} = \delta_{ab}\,\delta_{bd}\,P^{1,1;1,1}_{aa,aa}$, the first and the last labels of any n-point correlator coincide, i.e., we can associate with it a closed path $\{a, a_1, ...a_{n-1}, a\}$ with elements marked by the graph indices and passing through the coordinate points $z_1\,,\ldots,z_n$. E.g., the 2-point correlator reads

$$\langle {}^a\Psi^c_{j^*,\beta'}(z_1)\;{}^c\Psi^a_{j,\beta}(z_2)\rangle_a = {}^{(1)}F_{c1}\begin{bmatrix} j^* & j \\ a & a\end{bmatrix}^{1\,1}_{\beta'\,\beta} P_a\;\langle 0|\,\phi^1_{j^*j}(z_1)\;\phi^j_{j1}(z_2)\,|0\rangle\,. \tag{4.5}$$

For real arguments one recovers the correlators of the boundary fields. Note that the normalisation of boundary field correlators following from (3.1) differs from that used in [3], (eq. (4.7)) by a factor $\psi^1_1/\sqrt{S_{11}}$, i.e., $\langle \mathbf{1}\rangle_a = \frac{S_{11}}{(\psi^1_1)^2}\,\lim_{L/T\to\infty} Z_{1|a}\,e^{-\frac{\pi c}{6}\frac{L}{T}} = P_a$.

The algebra $\mathcal{A}$ acts on the operators (4.2) with the help of the antipode (3.21), namely for $\triangle(e^{(c'a')(c''a'')}_{p;\beta',\beta''}) = e_{(1)}\otimes e_{(2)}$ we define a representation $\tau(e)$

$$\begin{aligned} \tau(e^{(c'a')(c''a'')}_{p;\beta',\beta''})\;{}^c\Psi^a_{i,\beta}(z) &:= e_{(1)}\;{}^c\Psi^a_{i,\beta}(z)\,S(e_{(2)}) \\ &= \delta_{ip}\,\delta_{aa''}\,\delta_{cc''}\,\delta_{\beta\beta''}\,\left(\frac{P_{a'}\,P_c}{P_a\,P_{c'}}\right)^{\frac14}\;{}^{c'}\Psi^{a'}_{i,\beta'}(z)\,. \end{aligned} \tag{4.6}$$

Definition (4.2) has to be compared with earlier work [22,23] based on the use of quantum groups (Hopf algebras), or some related versions, e.g. [24], obtained by modifying the standard Hopf algebra axioms (see [4] for a discussion on the latter and further references). The papers [22,23,24] deal essentially with the diagonal case, and, more importantly, exploit the true $U_q(\mathfrak{g})$ $3j$-symbols at roots of unity (e.g., for $\mathfrak{g} = sl(2)$) in formulae analogous to (4.2). We stress again that the $3j$-symbols ${}^{(1)}F$ in (4.2) and (3.2) reduce in the diagonal case to the $6j$-symbols of the quantum groups $U_q(\overline{\mathfrak{g}})$ (or products of them), restricted to labels consistent with the CFT fusion rules. Thus the decomposition of the Ocneanu horizontal product fits precisely the CFT fusion, without the need of additional truncation as in the case of quantum group representations. As emphasized in [4], unlike the alternative approaches which deviate from the standard Hopf algebras, the use of a WHA as a quantum symmetry retains coassociativity reflected in (3.7). A "price" to be paid is the multiplicity of vacua, which has, however, a physical interpretation in BCFT, as providing a complete set of conformal boundary states.

From the operator representation (4.2) one derives various identities. In particular inserting the r.h.s. of (4.2) in the product of two generalised CVO, then applying the OPE for the standard CVO and finally using the pentagon identity (3.7), and once again the representation (4.2), we *derive* for small z_{12} the OPE

$$\begin{aligned} {}^{c}\Psi^{b}_{i,\alpha}(z_1)\,{}^{b}\Psi^{a}_{j,\gamma}(z_2) &= \sum_{p\,,\beta\,,t} {}^{(1)}F_{bp}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta\ \ t}_{\alpha\,\gamma} \sum_{\Pi} \langle p,\Pi|\phi^{p}_{ij;t}(z_{12})|j,0\rangle\ {}^{c}\Psi^{a}_{p,\beta;\Pi}(z_2) \\ &= \sum_{p,\beta,t} {}^{(1)}F_{bp}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\beta\ \ t}_{\alpha\,\gamma} \langle p,0|\phi^{p}_{ij;t}(z_{12})|j,0\rangle\ {}^{c}\Psi^{a}_{p,\beta}(z_2) + \ \ldots\,. \end{aligned} \tag{4.7}$$

For arguments restricted to the real line one recovers the boundary field small distance expansion [25] with OPE coefficients given by the $3j$-symbols of (3.2). Conversely, the expansion (4.7) was the starting point in [3] for the derivation of the pentagon identity (3.7).

Denote by ${}^{c}\mathcal{U}^{a}_{j}$ the space of generalised CVO (4.2). The generalised CVO have a nontrivial braiding defined through a new braiding matrix with $4+2$ indices of two types,

$$\hat{B}(\epsilon) : \oplus_b\, {}^{c}\mathcal{U}^{b}_{i} \otimes {}^{b}\mathcal{U}^{a}_{j} \to \oplus_d\, {}^{c}\mathcal{U}^{d}_{j} \otimes {}^{d}\mathcal{U}^{a}_{i}\,, \tag{4.8}$$

$${}^{c}\Psi^{b}_{i,\alpha}(z_1)\,{}^{b}\Psi^{a}_{j,\gamma}(z_2) = \sum_{d,\alpha',\gamma'} \hat{B}_{bd}\begin{bmatrix} i & j \\ c & a \end{bmatrix}^{\alpha'\,\gamma'}_{\alpha\,\gamma}(\epsilon)\ {}^{c}\Psi^{d}_{j,\alpha'}(z_2)\,{}^{d}\Psi^{a}_{i,\gamma'}(z_1)\,, \tag{4.9}$$

$$\hat{B}^{12}(\epsilon)\,\hat{B}^{21}(-\epsilon) = 1\,, \tag{4.10}$$

consistently with the commutativity of the intertwiners

$$\sum_b n_{ib}{}^c\, n_{ja}{}^b = \sum_d n_{jd}{}^c\, n_{ia}{}^d\,. \tag{4.11}$$

In (4.9) $z_{12} \notin \mathbb{R}_-$, and ϵ stands for $\epsilon_{12} = \mathrm{sign}(\mathrm{Im}(z_{12}))$ and for $i = 1$ or $j = 1$ the matrix $\hat{B}$ is trivial. The braiding matrices $\hat{B}$ satisfy the "Yang-Baxter (YB) equation"

$$\hat{B}^{12}(\epsilon_{12})\,\hat{B}^{23}(\epsilon_{13})\,\hat{B}^{12}(\epsilon_{23}) = \hat{B}^{23}(\epsilon_{23})\,\hat{B}^{12}(\epsilon_{13})\,\hat{B}^{23}(\epsilon_{12})\,. \tag{4.12}$$

Combining (4.9) with the definition (4.2) of the generalised CVO, using then the braiding of the standard CVO and projecting on the state $|0\rangle$, we obtain the relation

$$\sum_{d\,,\alpha'\,\gamma'} \hat{B}_{bd} \begin{bmatrix} i & j \\ c & a \end{bmatrix}_{\alpha\,\gamma}^{\alpha'\,\gamma'}\!(\epsilon)\ {}^{(1)}F_{dk} \begin{bmatrix} j & i \\ c & a \end{bmatrix}_{\alpha'\,\gamma'}^{\beta\ t} = e^{-i\pi\epsilon\,\triangle_{ij}^{k}}\ {}^{(1)}F_{bk} \begin{bmatrix} i & j \\ c & a \end{bmatrix}_{\alpha\,\gamma}^{\beta\,t}\,, \tag{4.13}$$

where the phase in the r.h.s., depending on the scaling dimensions $\triangle_{ij}^{k} = \triangle_i + \triangle_j - \triangle_k$, comes from the standard CVO braiding matrix B. In the diagonal case, where we can identify ${}^{(1)}F$ and $\hat{B}$ with the standard fusing and braiding matrices, F and B , this relation is nothing else than the simplest hexagon relation (the q-Racah identity). Inverting (4.13) we get a bilinear representation of $\hat{B}$ in terms of ${}^{(1)}F$

$$\hat{B}_{bd} \begin{bmatrix} i & j \\ c & a \end{bmatrix}_{\alpha\,\gamma}^{\alpha'\,\gamma'}\!(\epsilon) = \sum_{k\,,\beta\,,t} {}^{(1)}F_{bk} \begin{bmatrix} i & j \\ c & a \end{bmatrix}_{\alpha\,\gamma}^{\beta\ t}\ e^{-i\pi\epsilon\,\triangle_{ij}^{k}}\ {}^{(1)}F^{*}_{dk} \begin{bmatrix} j & i \\ c & a \end{bmatrix}_{\alpha'\,\gamma'}^{\beta\ t}\,. \tag{4.14}$$

This formula determines $\hat{B}$ whenever we know ${}^{(1)}F$ and the scaling dimensions $\triangle_j$, i.e., the $3j$-symbols ${}^{(1)}F_{bi}$ diagonalise the matrix $\hat{B}_{bd}$. It also implies the symmetries

$$\hat{B}_{bd} \begin{bmatrix} j & k \\ c & a \end{bmatrix}(\epsilon) = \hat{B}_{b^*d^*} \begin{bmatrix} k & j \\ a^* & c^* \end{bmatrix}(\epsilon) = \hat{B}_{db} \begin{bmatrix} j^* & k^* \\ a & c \end{bmatrix}(\epsilon) = \hat{B}^{*}_{b^*d^*} \begin{bmatrix} j^* & k^* \\ c^* & a^* \end{bmatrix}(-\epsilon)\,. \tag{4.15}$$

The relation (4.13) is a particular case of the more general identity derived by inserting (4.2) in (4.9) and using the analog of (4.9) for the standard CVO

$${}^{(1)}F\,{}^{(1)}F\,B = \hat{B}\,{}^{(1)}F\,{}^{(1)}F\,,$$

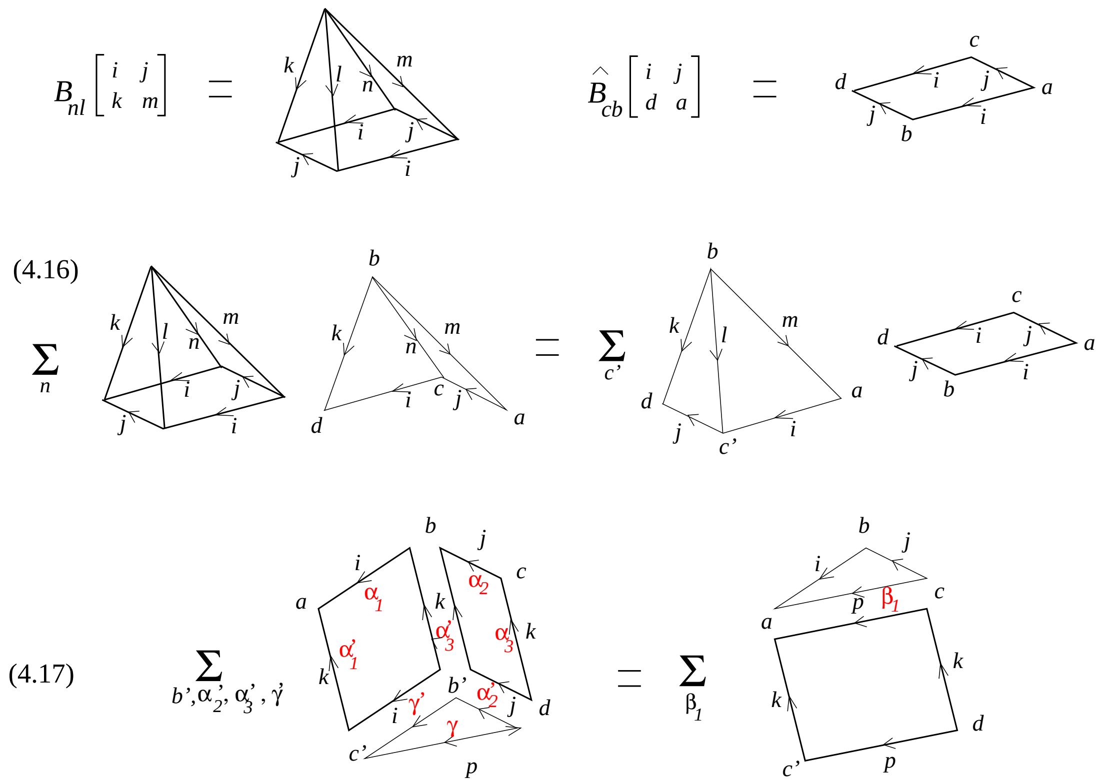

Fig. 7: Equations (4.16)-(4.17).

or, more explicitly

$$\begin{aligned}&\sum_{n\in\mathcal{I}} {}^{(1)}F_{an}\begin{bmatrix} j & m\\ c & b\end{bmatrix}\ {}^{(1)}F_{ck}\begin{bmatrix} i & n\\ d & b\end{bmatrix}\ B_{nl}\begin{bmatrix} i & j\\ k & m\end{bmatrix}\\ &=\sum_{c'\in\mathcal{V}} \hat{B}_{cc'}\begin{bmatrix} i & j\\ d & a\end{bmatrix}\ {}^{(1)}F_{al}\begin{bmatrix} i & m\\ c' & b\end{bmatrix}\ {}^{(1)}F_{c'k}\begin{bmatrix} j & l\\ d & b\end{bmatrix},\end{aligned}\tag{4.16}$$

as illustrated on Fig. 7. For $m=1$ we recover (4.13). Eq. (4.16) implies that (products of two) $3j$ symbols ${}^{(1)}F$ intertwine the two representations B and $\hat{B}$ of the braiding group. This is to be compared with the "cells" introduced in lattice models, [26,27,14,28], see section 6.

Another relation derived from the product of three GCVO Ψ gives a generalisation of the braiding-fusing identity of Moore-Seiberg [21]

$$\hat{B}\ {}^{(1)}F = \hat{B}\ \hat{B}\ {}^{(1)}F\,,$$

or,

$$\sum_{\beta_1} \hat{B}_{cc'}\begin{bmatrix} p & k\\ a & d\end{bmatrix}^{\alpha_1'\,\gamma}_{\beta_1\,\alpha_3}\ {}^{(1)}F_{bp}\begin{bmatrix} i & j\\ a & c\end{bmatrix}^{\beta_1\,t}_{\alpha_1\,\alpha_2} \tag{4.17}$$

$$= \sum_{b',\alpha_2',\alpha_3',\gamma'} {}^{(1)}F_{b'p} \begin{bmatrix} i & j \\ c' & d \end{bmatrix}_{\gamma' \, \alpha_2'}^{\gamma \, t} \hat{B}_{bc'} \begin{bmatrix} i & k \\ a & b' \end{bmatrix}_{\alpha_1 \, \alpha_3'}^{\alpha_1' \, \gamma'} \hat{B}_{cb'} \begin{bmatrix} j & k \\ b & d \end{bmatrix}_{\alpha_2 \, \alpha_3}^{\alpha_3' \, \alpha_2'}$$

In the diagonal case this is the equation from which one obtains (taking $d = 1$) the relation between the braiding B and fusing F matrices; inserting this relation back in (4.17) reproduces the standard pentagon identity for F. In general (4.17) provides a recursive construction of $\hat{B}$ in the spirit of [29]. Namely, solving it for the $\hat{B}$ matrix in the l.h.s., i.e., writing it as $\hat{B} = \sum_{b,b'} {}^{(1)}F \, \hat{B} \, \hat{B} \, {}^{(1)}F^*$, we get an equation which determines $\hat{B}$ recursively, given the subset of $3j$-symbols with one of the labels i, j, p fixed to the fundamental representation(s). Using (4.14) the r.h.s of (4.17) can be completed to $\hat{B}_{23} \, \hat{B}_{12} \, \hat{B}_{23}$, i.e., to the r.h.s. of the YB equation (4.12). Similarly one derives a second braiding - fusing identity so that its r.h.s. is completed to the l.h.s. of (4.12). Comparing the two identities and using twice in the l.h.s. of one of them the pentagon identity (3.7) and the unitarity of F, one recovers the YB relation.

Together with the interpretation of ${}^{(1)}F$ as $3j$-symbols, the braiding matrix $\hat{B}$ can be interpreted as the $\mathcal{R}$ – matrix of a quasitriangular WHA, see Appendix A. The relation (4.14) is an analogue of the representation of the $\mathcal{R}$ matrix in terms of the $3j$-symbols, while (4.16) is an analogue of the relation between vertex representation and path representation of the quantum group $\mathcal{R}$ matrix (Vertex -IRF correspondence), see [30].

There is an important difference in the analogy with the rôle of quantum groups in CFT. Namely in the present approach the summations in all identities, like e.g., over k in (4.14), or over n in (4.16), run according to the fusion rules, while in their analogues, where the true quantum group $3j$-symbols appear, these summations run within the standard classical tensor product bounds. When interpreted in the CFT framework the analogue of the braiding relation (4.9) is then required to hold only on a "physical" subspace, or alternatively, the conformal Hilbert spaces (and the conventional CVO in the definition of the covariant CVO of [22]) have to be extended to accomodate "unphysical" intermediate states incompatible with the fusion rules, [31,23,32], see also the recent work [33] for a related discussion and further references.

5. Bulk fields – chiral representation

Let now the pairs $I = (i, \bar{i})$, $i, \bar{i} \in \mathcal{I}$, label the "physical" spectrum, corresponding to nonzero matrix elements of the modular mass matrix $Z_{i\bar{i}}$, or, in a more precise notation,

which we will for simplicity skip in this section, $(i,\bar{i};\alpha)\,, \alpha=1,2\ldots Z_{i\bar{i}}$. We define (upper) half-plane bulk fields as compositions of two GCVO (4.2)

$$\begin{aligned}\Phi^H_{(i,\bar{i})}(z,\bar{z}) &= \sum_{a,b\,,\beta'\,,\beta} \Big(\sum_{j,\alpha,u} R^{(i,\bar{i}^*,u)}_{a,\alpha}(j)\;{}^{(1)}F^*_{bj}\begin{bmatrix} i & \bar{i}^* \\ a & a\end{bmatrix}^{\alpha\, u}_{\beta\,\beta'}\Big)\;{}^{a}\Psi^{b}_{i,\beta}(z)\;\;{}^{b}\Psi^{a}_{\bar{i}^*,\beta'}(\bar{z}) \\ &= \sum_{n\,,k\,,l\,,t\,,t'} \phi^n_{ik;t'}(z)\,\phi^k_{\bar{i}^*l;t}(\bar{z})\otimes \sum_{a\,,b'\,,\gamma\,,\gamma'} C^{n,k,l;t',t}_{(i,\bar{i})\,a,b',a;\gamma,\gamma'}\;P^{n,\gamma;l,\gamma'}_{ab',ab'}\,. \end{aligned} \tag{5.1}$$

Here $\bar{z}\in H_-$ is the complex conjugate of $z\in H_+$ and the field $\Phi^H_{(i,\bar{i})}(z,\bar{z})$ transforms under a tensor product representation of one copy of the chiral algebra $\mathfrak{A}$ labelled by the pairs $(i,\bar{i}^*)$, see [34,35,3] for discussions of more general gluing conditions. [3] The choice of the constants in (5.1), related according to

$$C^{n,k,l;t',t}_{(i,\bar{i})\,a,b',a;\gamma,\alpha} = \sum_{j\,,u\,,u'\,,\alpha'} R^{(i,\bar{i}^*,u)}_{a,\alpha'}(j)\;\;{}^{(1)}F_{an}\begin{bmatrix} j & l \\ a & b'\end{bmatrix}^{\gamma\, u'}_{\alpha'\,\alpha} F^*_{kj}\begin{bmatrix} i & \bar{i}^* \\ n & l\end{bmatrix}^{u'\, u}_{t'\,t}\,, \tag{5.2}$$

is such that when applying for small $z-\bar{z}=2i\,y$ the OPE (4.7) for the two CVO in (5.1) (and projecting on $|e^1_{aa}\rangle$) we recover in the leading order the boundary field ${}^{a}\Psi^{a}_{j,\alpha}(x)$ contributing with the *bulk-boundary reflection coefficient* $R^{(i,\bar{i}^*,u)}_{a,\alpha}(j)=C^{j,\bar{i}^*,1;u,1}_{(i,\bar{i})\,a,a,a;\alpha,1}$ of [25]. (We denote here $R^{(i,\bar{i},u)}_{a,\alpha}(j)$ what was denoted ${}^{a;\alpha}B^{j;u}_{(i,\bar{i})}$ in [3].) For $j=1$ it is expressed in terms of the graph eigenvector matrices ψ^i_a

$$R^{(i,i^*)}_a(1)=\frac{\psi^i_a}{\psi^1_a}\;\frac{e^{i\pi\triangle_i}}{\sqrt{d_i}}\,. \tag{5.3}$$

From the operator representations (4.2) and (5.1), which involve the two sets of constants, ${}^{(1)}F$ and R (or, C), one recovers all correlators of the fields Ψ and Φ^H; they are expressed as linear combinations of standard CVO correlators. E.g., the 2-point function projected on the state $|e^1_{aa}\rangle$, is

$$\langle\,{}^{c}\Psi^{b}_{j,\alpha}(z_2)\Phi^H_I(z,\bar{z})\rangle_a\;=\delta_{ab}\,\delta_{ac}\,\frac{P_a}{\sqrt{d_j}}\sum_t R^{(i,\bar{i}^*,t)}_{a,\alpha^+}(j^*)\,\langle 0|\phi^1_{jj^*}(z_2)\,\phi^{j^*}_{i\bar{i}^*;t}(z)\,\phi^{\bar{i}^*}_{\bar{i}^*1}(\bar{z})\,|0\rangle\,. \tag{5.4}$$

[3] For convenience we keep the same notation for the half-plane field $\Phi^H_{(i,\bar{i})}$ (with $(i,\bar{i}^*)$ appearing in the CVO product in the r.h.s. of (5.1)) as for its full-plane counterpart $\Phi^P_{(i,\bar{i})}$, with the second label in $(i,\bar{i})$ corresponding to a representation of a second copy of the chiral algebra; in our convention the diagonal torus modular invariants correspond to the fields $\Phi^H_{(i,i)}$, $i\in\mathcal{I}$.

In (5.4) we have adopted an ordering corresponding to real parts increasing from right to left, i.e., Re $(z_2 - z) > 0$. The inverse order would give a function which differs by a phase (due to the nontrivial braiding of products of CVOs), even if the difference of scaling dimensions, the spin $s_I = \triangle_i - \triangle_{\bar{i}}$, is (half)integer, as required from the physical spectrum. The phase vanishes if we furthermore restrict the argument z_2 of the generalised CVO to the real axis boundary of H_+ and thus the bulk and the boundary fields commute.

Let us now briefly review the derivation of the equations resulting from the sewing constraints of Cardy–Lewellen [25,36] in the BCFT. The operator representations introduced here both for the boundary and the bulk fields make these derivations straightforward (in fact also slightly more general) and reduce them to the use of the fusing and braiding relations for the conventional CVO. First requiring locality (commutativity) of a bulk and boundary operators, $\Phi_I^H(z,\bar{z})\, {}^a\Psi_j^b(x_2) = {}^a\Psi_j^b(x_2)\, \Phi_I^H(z,\bar{z})$, has further implications, leading to an equation for the unknown constant C in the operator representation (5.1). It reads, omitting for simplicity the multiplicity indices

$$\begin{aligned} &\sum_l C^{n,k,l}_{(i,\bar{i})\,a,b',a} \quad {}^{(1)}F_{bl} \begin{bmatrix} j & g \\ a & b' \end{bmatrix} \; B_{ll'} \begin{bmatrix} \bar{i}^* & j \\ k & g \end{bmatrix} (-) = \\ &\sum_{k'} C^{k',l',g}_{(i,\bar{i})\,b,b',b} \quad {}^{(1)}F_{bn} \begin{bmatrix} j & k' \\ a & b' \end{bmatrix} \; B_{k'k} \begin{bmatrix} j & i \\ n & l' \end{bmatrix} (-)\,. \end{aligned} \tag{5.5}$$

Projecting the product of two fields on $|0\rangle$, or on $\langle 0|$, i.e., setting $g = 1$, or $k = 1$ in (5.5), one recovers the (first) bulk-boundary Cardy-Lewellen equation [25,36]; (5.5) is a slightly more general version of it, corresponding to a 5-point chiral block. This equation provides a closed expression for the scalar reflection coefficients $R^{(i,i^*,t)}_{a,\alpha}(j)$ in terms of the $3j$-symbols ${}^{(1)}F$ and the modular matrix $S(j)$ of 1-point torus correlators

$$\frac{P_a}{\sqrt{d_j}}\, \frac{R^{(i,i^*)}_{a,\alpha}(j^*)}{R^{(i,i^*)}_1(1)} = S_{i1} \sum_{k,b,\beta} \frac{\psi^i_b}{\psi^i_1}\; {}^{(1)}F_{ak} \begin{bmatrix} k & j \\ b & a \end{bmatrix}^{\beta}_{\beta\,\alpha} \; S_{ki^*}(j)\,. \tag{5.6}$$

In the diagonal case the l.h.s. reproduces $S_{ai}(j)/S_{1i}$ [5,3].

With (5.1) at hand one also *derives* the OPE of two bulk fields $\Phi_K^H\, \Phi_L^H$ first expressing their product as a product of four standard CVO, exchanging then the second and third fields and fusing each of the two pairs labelled by (k,l) and $(\bar{k}^*, \bar{l}^*)$ (this can be depicted by a 6-point chiral block diagram slightly more general than Fig. 10 of [3]). In the process one finds an expression for the OPE coefficients, to be denoted $d_{K\,L}^{J;t,t'}$. It reads symbolically, ordering the constants in the l.h.s. in the sequence they appear in the above steps,

$$F\,F\,B(-)\,C\,C = d\,C\,, \tag{5.7}$$

or

$$
\begin{aligned}
& C^{m,n,g'}_{(k,\bar{k})\,a,b,a}\; C^{g',\bar{n},i}_{(l,\bar{l})\,a,b,a} = \\
& \sum_{j,\bar{j},g} d^{J}_{K\,L}\, B_{gg'}\begin{bmatrix} l & \bar{k}^* \\ n & \bar{n}\end{bmatrix}(+)\; F^*_{nj}\begin{bmatrix} k & l \\ m & g\end{bmatrix}\; F^*_{\bar{n}\bar{j}^*}\begin{bmatrix} \bar{k}^* & \bar{l}^* \\ g & i\end{bmatrix}\; C^{m,g,i}_{(j,\bar{j})\,a,b,a}\,.
\end{aligned}
\tag{5.8}
$$

Setting $i=1$ and substituting the constants C with the reflection coefficients R as in (5.2), this can be also rewritten, introducing a new constant M, as

$$
R^{(k,\bar{k};u_1)}_{a,\alpha_1}(p_1)\; R^{(l,\bar{l};u_2)}_{a,\alpha_2}(p_2) = \sum_{j,\bar{j},p_3,u_3,\alpha_3} M_{(k,\bar{k},u_1;p_1,\alpha_1))(l,\bar{l},u_2;p_2,\alpha_2)}^{\quad\;(j,\bar{j},u_3;p_3,\alpha_3);a}\; R^{(j,\bar{j};u_3)}_{a,\alpha_3}(p_3) \tag{5.9}
$$

with $u_1 = 1,\ldots, N_{k\bar{k}}{}^{p_1}$, $\alpha_1 = 1,\ldots, n_{p_1 a}{}^{a}$, etc. This is the second of the two basic bulk-boundary Lewellen equations [36]. In the diagonal case $K=(k,k)$ the OPE coefficients $d=d^{(H)}$ coincide with their full-plane counterparts $d^{(P)}$ and in the unitary gauge used here are simply $d^{(P)\,J;t,t'}_{K\,L} = \delta_{tt'}$ for $N_{kl}{}^{j}\neq 0$.

The equation (5.9), taken at $p_1=p_2=1$, allows to derive and generalise to higher rank cases (see [37,3]) the empirical $sl(2)$ result of [17] on the coincidence of the relative scalar OPE coefficients and the structure constants of the Pasquier algebra [16]. The latter algebra has 1-dimensional representations (characters) given by $\frac{\psi^i_a}{\psi^1_a} = e^{-\pi i \triangle_i}\sqrt{d_i}\, R^{(i,i^*)}_a(1)$, cf. (5.3). A generalisation of this result will be discussed in section 7 below.

The reflection coefficients satisfy

$$
(R^{(i,\bar{i})}_a(j))^* = R^{(i,\bar{i})}_{a^*}(j^*)\, e^{-i\pi\triangle^{j}_{i\bar{i}}} = R^{(i^*,\bar{i}^*)}_a(j^*)\, e^{-i\pi\triangle^{j}_{i\bar{i}}}\,, \tag{5.10}
$$

and furthermore (5.8) implies, choosing (the positive) constant $d^1_{K\,K^*}=1$.

$$
\sum_{a\,,\alpha} R^{(k,\bar{k};u)}_{a,\alpha}(j)\; R^{(l,\bar{l};u')\,*}_{a,\alpha}(j)\; (\psi^1_a)^2\; \frac{d_k}{d_j} = \delta_{lk}\,\delta_{\bar{l}\bar{k}}\,\delta_{uu'}\,\delta_{\bar{k}k^*}\,. \tag{5.11}
$$

The identity (5.11) reduces for $j=1$ to the orthonormality property of ψ^l_a (expressing the completeness of the set of boundary states) and in the diagonal cases to the unitarity relation for the modular matrices $S(j)$. In general $d^{(H)}$ and $d^{(P)}$ differ by phases depending on the spins $s_K = \triangle_k - \triangle_{\bar{k}}$,

$$
d^J_{K\,L} = e^{-i\frac{\pi}{2}(s_K+s_L-s_J)}\, d^{(P)\,J}_{K\,L}\,, \tag{5.12}
$$

reproducing in particular the spin-dependent full-plane 2-point function normalisation, $d^{(P)\,1}_{K\,K^*} = (-1)^{s_K}$, proved to be consistent with the locality and reflection positivity requirements [38].

6. Relation to integrable lattice models

Some of the identities in sections 3 and 4, most notably the YB equation, coincide with the basic identities of the related IRF integrable lattice models. The lattice Boltzmann weights, however, depend on a spectral parameter u, which does not appear in the CFT, and to compare the two discussions, a proper limit of this parameter has to be taken. This correspondence has been established in the diagonal cases, [39], and in this section we show how it generalises to all models built on graphs related to $\widehat{sl}(n)_{h-n}$ CFT.

The data required to define the generalised $sl(n)$-IRF models that we consider are a graph G – we postulate that one of the graphs met in the CFT discussion is appropriate– and a pair of representations j_1 and j_2 for $sl(n)$. Then to each vertex of the square lattice is assigned a vertex a of the graph. The Boltzmann weights $W_{j_1 j_2}\left(\begin{smallmatrix} c & d \\ b & a \end{smallmatrix}\right)(u)$ are functions of the four vertices a, b, c, d around a square face and of a spectral parameter u. It is conventional to tilt the lattice by 45 degrees and to represent the Boltzmann weights as in Fig. 8. Representation j_1 is assigned to the SW-NE bonds, and j_2 to the SE-NW ones [40]. Intuitively, one goes from vertex a to vertex b through the action of j_2, and from b to c through j_1, and accordingly, the weights depend also in general on bond labels $\alpha, \gamma, \cdots$, which specify which path from a to b, from b to c etc is chosen, : $\alpha = 1, \cdots, {n_{j_1\, b}}^c$, $\gamma = 1, \cdots, {n_{j_2\, a}}^b$, etc.

The Boltzmann weights are solutions of the spectral parameter dependent YB and inversion (unitarity) equations. Knowing them for the fundamental representation(s) enables one to construct the other weights by a fusion procedure [41,42,43].

In the simplest case where implicitly all the bonds carry the fundamental representation $\square$ of $sl(n)$, the Boltzmann weights have the general form

$$W \begin{pmatrix} c & d \\ b & a \end{pmatrix}_{\alpha\,\gamma}^{\alpha'\,\gamma'} (u) = \sin(\frac{\pi}{h} - u)\delta_{bd} + \sin(u)\, [2]_q\, U \begin{bmatrix} c \\ a \end{bmatrix}_{b\,\alpha\,\gamma}^{d\,\alpha'\gamma'} , \tag{6.1}$$

where $[2]_q = 2\cos(\frac{\pi}{h})$ for $q = e^{-2\pi i \frac{h-1}{h}}$, (h the Coxeter number of the graph G), and $[2]_q\, U$ are Hecke algebra generators satisfying $U^2 = U$ etc. Choosing the labels $j = k = \square$ in the bilinear representation (4.14) for the braiding matrix $\hat{B}$, we can cast it into a form similar to (6.1)

$$\hat{B}_{bd}(\varepsilon) = \delta_{bd}\, q^{a\varepsilon} \; - q^{b\varepsilon}\; C\, U_{bd} \,, \tag{6.2}$$

with (cf. Fig. 8)

$$U \begin{bmatrix} c \\ a \end{bmatrix}_{b\,\alpha\gamma}^{d\,\alpha'\gamma'} = \sum_{\beta} {}^{(1)}F_{b\,\begin{smallmatrix}\square\\\square\end{smallmatrix}} \begin{bmatrix} \square & \square \\ c & a \end{bmatrix}_{\alpha\,\gamma}^{\beta\,1} {}^{(1)}F^*_{d\,\begin{smallmatrix}\square\\\square\end{smallmatrix}} \begin{bmatrix} \square & \square \\ c & a \end{bmatrix}_{\alpha'\,\gamma'}^{\beta\,1} . \tag{6.3}$$

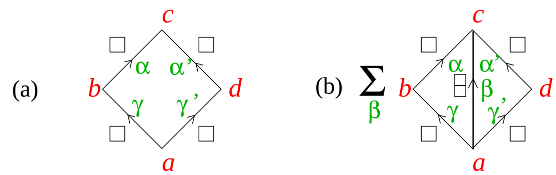

Fig. 8: (a): the Boltzmann weight W for $j_1 = j_2 = \square$; (b) U as a product of two cells

The constants a, b, C are determined from (4.10) and (4.13); from (4.10) we get $C = q^{a-b} + q^{b-a}$, and from (4.13) we get $a = -h(\triangle^{\rm Su}_{\square} - \frac{1}{2}\triangle^{\rm Su}_{\square\square}) \, , 2b - a = -h(\triangle^{\rm Su}_{\square} - \frac{1}{2}\triangle^{\rm Su}_{\boxminus}) \, ,$ hence $C = [2]_q$. Here $\triangle^{\rm Su}_{\lambda}$ are Sugawara conformal dimensions, while in (4.13) enter the dimensions $\triangle_{\lambda} = \triangle^{\rm Su}_{\lambda} - <\lambda, \rho>$ of the minimal W_n model of central charge $c = (n-1)[1 + 2n(n+1) - n(n+1)(\frac{h-1}{h} + \frac{h}{h-1})]$; this shift of the dimensions is accounted for by the sign in front of the second term in (6.2). One obtains $a = \frac{n-1}{2n} \, , b = -\frac{1}{2n}$.

When (6.2) is inserted in the YB equation (4.12), the latter reduces to the Hecke algebra relation for the operators $[2]_q U$ in (6.3), which can be identified with the operators in the r.h.s. of (6.1). Thus the Hecke generators are expressed in terms of the $3j$-symbols ${}^{(1)}F$, recovering a formula in [2]. Furthermore comparing (6.1) and (6.2) we obtain

$$\hat{B}_{bd} \begin{bmatrix} \square & \square \\ c & a \end{bmatrix}^{\alpha' \, \gamma'}_{\alpha \, \gamma} (\epsilon) = 2i \, q^{-\epsilon \frac{1}{2n}} \lim_{u \to -i\epsilon\infty} e^{-i\pi\epsilon u} \, W \begin{pmatrix} c & d \\ b & a \end{pmatrix}^{\alpha' \, \gamma'}_{\alpha \, \gamma} (u) \, . \qquad (6.4)$$

In other words, we can look at the correlators of the generalised CVO with all representation labels fixed to the fundamental ones as realising a representation of the corresponding Hecke algebra – in parallel to the path representations of the lattice theory. In the $sl(2)$ case (6.2), (6.3) reproduce, inserting (3.11), the Boltzmann weight of the ADE Pasquier models [44]

$$q^{\frac{1}{4}} \, \hat{B}_{bd} \begin{bmatrix} f & f \\ a & c \end{bmatrix} = q^{\frac{1}{2}} \delta_{bd} - \delta_{ac} \, \frac{\sqrt{P_b P_d}}{P_a} \, . \qquad (6.5)$$

There is no general information on ${}^{(1)}F$ in the higher rank cases, however the particular (fundamental) matrix elements in (6.3) are recovered from the $sl(n)$ examples of Boltzmann weights found in the literature, [45,46,47,14,48]. Recently exhaustive results were obtained [2] for all $sl(3)$ graphs but one. A general existence theorem for ${}^{(1)}F$ and W for a subclass of graphs corresponding to conformal embeddings appears in [49]. On the other hand, as the counter-example of [2] shows, some solutions of (1.1) do not support a representation of the Hecke algebra, i.e., a system of $3j$-symbols ${}^{(1)}F$.

In the $sl(2)$ and $sl(3)$ cases one can formulate [2] a quartic relation directly for the cells ${}^{(1)}F$ which in turn implies the Hecke algebra (or YB) relations; in our notation it reads

$$\begin{aligned}
&\sum_{\substack{b', \\ \alpha_1,\alpha_2, \\ \alpha_3\, \alpha_4}} {}^{(1)}F^{*}_{b'\boxminus}\begin{bmatrix} \square & \square \\ a & c \end{bmatrix}^{\gamma_1}_{\alpha_1\alpha_2} {}^{(1)}F_{c\boxminus}\begin{bmatrix} \square & \square \\ b' & d \end{bmatrix}^{\alpha_3}_{\alpha_2\gamma_2} {}^{(1)}F^{*}_{c'\boxminus}\begin{bmatrix} \square & \square \\ b' & d \end{bmatrix}^{\alpha_3}_{\alpha_4\gamma_3} {}^{(1)}F_{b'\boxminus}\begin{bmatrix} \square & \square \\ a & c' \end{bmatrix}^{\gamma_4}_{\alpha_1\alpha_4} \\
&= \sum_{\substack{b', \\ \alpha_1,\alpha_2 \\ \alpha_3,\alpha_4}} \frac{\sqrt{P_c\, P_{c'}}}{P_{b'}} {}^{(1)}F^{*}_{b'\boxminus}\begin{bmatrix} \square & \square \\ a & c \end{bmatrix}^{\gamma_1\, 1}_{\alpha_1\alpha_2} {}^{(1)}F_{b'\square^*}\begin{bmatrix} \boxminus^{\,*} & \square \\ d & c \end{bmatrix}^{\gamma_2^+\, 1}_{\alpha_3\alpha_2} {}^{(1)}F^{*}_{b'\square^*}\begin{bmatrix} \boxminus^{\,*} & \square \\ d & c' \end{bmatrix}^{\gamma_3^+\, 1}_{\alpha_3\alpha_4} {}^{(1)}F_{b'\boxminus}\begin{bmatrix} \square & \square \\ a & c' \end{bmatrix}^{\gamma_4\, 1}_{\alpha_1\alpha_4} \\
&= \frac{1}{[2]^2}\Big(\frac{\sqrt{P_cP_{c'}}}{P_a}\ \delta_{ad}\,\delta_{\gamma_1\,\gamma_2}\,\delta_{\gamma_3\,\gamma_4} + \delta_{cc'}\,\delta_{\gamma_1\,\gamma_4}\,\delta_{\gamma_2\,\gamma_3}\Big)
\end{aligned} \tag{6.6}$$

The first delta-term here is present only for the $sl(3)$ case where $\boxminus^{\,*} = \square$ and the two 3-point couplings corresponding to the δ function are identical; for $n=2$, where by convention $\boxminus$ refers to the identity representation, the first term is zero; accordingly we recover the TLJ algebra relation.

The fused Boltzmann weights are similarly expected to be related to more general braiding matrix elements. The recursive construction of the general $\hat{B}$ elements using the fusing-braiding relation (4.17) is analogous to the fusion procedure of the lattice models yielding the fused Boltzmann weights. The "inversion equation" for the Boltzmann weights in the lattice models turns into the unitarity identity (4.10). The relation (4.17) taken for $p=1$ leads to the (crossing) identity

$$\sum_{b'} \hat{B}_{cb'}\begin{bmatrix} i^* & k \\ b & d \end{bmatrix}(\epsilon)\, \hat{B}_{bd}\begin{bmatrix} i & k \\ a & b' \end{bmatrix}(\epsilon) \sqrt{\frac{P_{b'}\, P_a}{P_b\, P_d}} = \delta_{ac}\,, \tag{6.7}$$

while (4.13) with $i=1$ reads

$$\sum_{d} \hat{B}_{bd}\begin{bmatrix} j & j^* \\ a & a \end{bmatrix}(\epsilon)\ \sqrt{P_d} = \sqrt{P_b}\, e^{-2\pi i \epsilon\, \triangle_j}\,, \tag{6.8}$$

a property analogous to one satisfied by the full (u - dependent) Boltzmann weights.

We now turn to the relation (4.16), which has the form of the intertwining relation for the square Ocneanu cells, studied in [26,14,27,28]. To make contact with the notation in [14], ${}^{(1)}F_{aj}\left[\begin{smallmatrix} \square & i \\ c & b \end{smallmatrix}\right]$ is identified with $Y\left[\begin{smallmatrix} a & i \\ c & j \end{smallmatrix}\right]$ with b fixed and i, j, c, a restricted by ${n_{ib}}^a\,, {n_{jb}}^c \neq 0$. The data found in those papers provide thus a partial information on the $3j$-symbols (i.e., on the boundary field OPE coefficients), namely determine those matrix elements in

which one of the representation labels is fixed to the fundamental weight $\square$ and b fixed to 1. On the other hand, knowledge of the cells ${}^{(1)}F_{aj}\left[\begin{smallmatrix}\square & i\\ c & b\end{smallmatrix}\right]$ for arbitrary a, b, c and i, j is sufficient to determine all the cells using the pentagon equation, in a way similar to the discussion at the end of section 4. A general solution for ${}^{(1)}F$ for the $\widehat{sl}(2)$ D-series has been found in [6].

We conclude with the remark that it would be interesting to extend the correspondences discussed in this section to the boundary lattice theories, see [50,51], and in particular to clarify the rôle of the reflection equations [52] in the present setting.

7. Ocneanu graphs and the associated algebras

In the following we shall motivate on physical grounds and by analogy with a situation already encountered in BCFT the construction of new sets of (non-negative integer valued) matrices and their associated graphs. On a mathematical level, this construction has been justified in the subfactor approach [1,2,10,11], but the field theoretical approach provides new insight.

In BCFT we know that three sets of matrices play an interlaced rôle, generalising the fusion matrices N_i. The first is the set of $|\mathcal{V}| \times |\mathcal{V}|$ matrices n_i defined in (1.1), which form a representation of the fusion algebra and define the graph G. As recalled in section 2, their diagonalisation introduces a set of orthonormal eigenvectors ψ_a^j, $a \in \mathcal{V}$, $j \in \mathrm{Exp}$.

The second set of matrices, also of size $|\mathcal{V}| \times |\mathcal{V}|$, denoted $\hat{N}_a = \{\hat{N}_{ba}{}^c\}$ in [3], forms the regular representation of an associative algebra,

$$\hat{N}_a \hat{N}_b = \hat{N}_{ab}{}^c \hat{N}_c \tag{7.1}$$

(the Ocneanu algebra) [53,26]. It is attached to the graph in the sense that

$$n_i \hat{N}_a = \sum_b n_{ia}{}^b \, \hat{N}_b \ , \tag{7.2}$$

i.e., if the matrix $\hat{N}_a$ is assigned to vertex a of the graph G, the action of n_i on $\hat{N}_a$ gives a sum over the neighbouring matrices $\hat{N}_b$ (neighbouring in the sense of the adjacency matrix $n_{ia}{}^b$).

In general, these matrices $\hat{N}_a$ have entries that are integers, but in general of indefinite sign [4]. At this point, we recall that RCFT and the associated graphs G come in two types. Those for which the modular invariant partition function is block-diagonal and expressible in terms of the n matrices as

$$Z_{ij} = \sum_{a \in T} n_{i1}{}^{a} n_{j1}{}^{a} \tag{7.3}$$

for a certain subset T of vertices are called of type I. They are interpreted as diagonal theories in the sense of some extended chiral algebra $\mathfrak{A}^{\rm ext}$. The set T is in one-to-one correspondence with the set of ordinary representations of that algebra $\mathfrak{A}^{\rm ext}$ and the integer $n_{i1}{}^{a}$ is the multiplicity $\mathrm{mult}_a(i)$ of representation $\mathcal{V}_i$ of $\mathfrak{A}$ in the representation of $\mathfrak{A}^{\rm ext}$ labelled by a. Then all matrices $\hat{N}_a$, $a \in \mathcal{V}$ have non negative integer entries and the subset $\{\hat{N}_a\}_{a\in T}$ forms a subalgebra isomorphic to the fusion algebra of $\mathfrak{A}^{\rm ext}$, [15][5]

An interpretation of the whole set of $\hat{N}_{ab}{}^{c}$ as fusion coefficients of a class of "twisted" representations of $\mathfrak{A}^{\rm ext}$ broader than considered in section 2 has been proposed in [3,55], see also [56] and sect 7.6 below. In contrast, a theory of type II cannot be written as in (7.3) and is obtained from some type I one –its "parent theory"– through an automorphism of its fusion rules acting on its right sector with respect to the left one [21,57]. We thus expect many of their properties to be more simply expressed in terms of data pertaining to the parent theory. For example, their torus partition function reads

$$Z_{ij} = \sum_{a \in T} n_{i1}{}^{a} n_{j1}{}^{\zeta(a)} \ , \tag{7.4}$$

where the n's are those of the parent type I theory.

We can then define the dual (in the sense of [58]) of the $\hat{N}$ algebra by the algebra of linear maps $\hat{N} \to \mathbb{C}$. This algebra, also called the Pasquier algebra, is realised by matrices $M_{(i,\alpha)}$ labelled by the elements of Exp and as mentioned in section 5 relates to the scalar OPE coefficients.

---

[4] A case where this integrality property of the $\hat{N}_{ab}{}^{c}$ seemed invalid was pointed out in [18], but later it was shown by Xu that integrality could be restored at the expense of commutativity [49], see below sect 7.2.

[5] These statements are for us empirical facts, of which we know no general proof. They seem to have been established for a variety of cases in the subfactor approach or are taken as assumptions. Note that our definition of type I in (7.3) above is slightly more restrictive than the one used previously [18,3]. It rules out one of the graphs ($\mathcal{E}_3^{(12)}$ in the Table of [3]). See also [54] for cases which go beyond this simple classification.

As a side remark, we recall that in the $sl(2)$ case it is this M algebra which also appears as the perturbed chiral ring of $N = 2$ superconformal CFTs perturbed by their least relevant operator (or of their topological counterparts) [59], hence as a specialisation of the Frobenius algebra [60]. We shall return to these algebras and their CFT interpretation in the next sections.

In the following, we are going to introduce four sets of matrices, which generalise the previous three, define again graph(s) $\widetilde{G}$, and satisfy analogous relations. The matrices n gives rise to two sets, denoted $\tilde{V}$ and $\tilde{n}$, while the dual pair $(\hat{N}, M)$ generalises to a pair $(\widetilde{N}, \widetilde{M})$.

7.1. The $\tilde{V}$ matrices and Ocneanu graphs

We first consider the integral, nonnegative matrix solutions of a system of equations for commuting matrices $\tilde{V}_{ii';\,x}{}^{y}$ with $i, i' \in \mathcal{I}$. It generalises (1.1), with the Verlinde fusion multiplicities $N_{ij}{}^{k}$ replaced by the product $N_{ij}{}^{k}\, N_{i'j'}{}^{k'}$

$$\sum_{y} \tilde{V}_{ij;\,x}{}^{y}\, \tilde{V}_{i'j';\,y}{}^{z} = \sum_{i'',j''} N_{ii'}{}^{i''}\, N_{jj'}{}^{j''}\, \tilde{V}_{i''j'';\,x}{}^{z}\,. \tag{7.5}$$

The labels $x, y, \cdots$ of these matrices take their values in a finite set denoted $\widetilde{\mathcal{V}}$, whose cardinality equals $|\widetilde{\mathcal{V}}| = \sum_{j\bar{j}} (Z_{j\bar{j}})^2$ in terms of the modular invariant matrix Z.

This property, and more generally the physical interpretation of (7.5), follow from the discussion of torus partition functions in the presence of twist operators (physically defect lines) denoted X_x, see [8] for details. The discussion is parallel to the way equation (1.1) appears in the study of cylinder partition functions and involves the consistency between two alternative pictures. In one picture, two twist operators $X_x^\dagger$ and X_y, attached to homology cycles of type a of the torus, act in the Hilbert space of the ordinary bulk theory, $\mathcal{H} = \oplus Z_{i\bar{i}}\ \mathcal{V}_i \otimes \bar{\mathcal{V}}_{\bar{i}}$, and are assumed to commute with the generators of the two copies of the chiral algebra $\mathfrak{A}$. This forces them to be linear combinations of operators $P^{(k,\bar{k};\gamma,\gamma')}$ intertwining the different copies of equivalent representations of $\mathfrak{A} \times \mathfrak{A}$

$$X_x = \sum_{\substack{i,\bar{i} \\ \alpha,\alpha'=1,\cdots Z_{i\bar{i}}}} \frac{\Psi_x^{(i,\bar{i};\alpha,\alpha')}}{\sqrt{S_{1i} S_{1\bar{i}}}}\ P^{(i,\bar{i};\alpha,\alpha')}\ , \tag{7.6}$$

with

$$P^{(i,\bar{i};\alpha,\alpha')}\, P^{(j,\bar{j};\beta,\beta')} = \delta_{ij} \delta_{\bar{i}\bar{j}} \delta_{\alpha'\beta}\, P^{(i,\bar{i};\alpha,\beta')}\ . \tag{7.7}$$

The other picture makes use of a Hilbert space $\mathcal{H}_{x|y}$ associated with the homology cycles of type $\mathtt{b}$; the non-negative integer $\tilde{V}_{ij^*;\,x}{}^y$ describes the multiplicity of representation $\mathcal{V}_i \otimes \overline{\mathcal{V}_j}$ in $\mathcal{H}_{x|y}$. The equality of the twisted partition functions computed in these two alternative ways leads to a consistency condition of the form

$$\tilde{V}_{i\bar{i};\,x}{}^y = \sum_{\substack{J \\ \alpha,\alpha'=1,\cdots Z_{j\bar{j}}}} \frac{S_{ij}S_{\bar{i}\bar{j}}}{S_{1j}S_{1\bar{j}}}\ \Psi_x^{(j,\bar{j};\alpha,\alpha')}\ \Psi_y^{(j,\bar{j};\alpha,\alpha')\,*}\,, \qquad i,\bar{i} \in \mathcal{I}\,, \tag{7.8}$$

where $\Psi_y^{(j,\bar{j};\alpha,\alpha')\,*}$ is the complex conjugate of $\Psi_y^{(j,\bar{j};\alpha,\alpha')}$. Then $\Psi = \{\Psi_x^{(J;\alpha,\beta)}\}$ is assumed to be a square, unitary matrix, labelled by the $x \in \widetilde{\mathcal{V}}$ and by the pairs $J = (j, \bar{j})$ of labels supplemented by their multiplicities in the spectrum $\alpha, \beta = 1, \cdots, Z_{j\bar{j}}$

$$\begin{aligned} \sum_{x\in\widetilde{\mathcal{V}}} \Psi_x^{(J;\alpha,\beta)}\Psi_x^{(J';\alpha',\beta')\,*} &= \delta_{jj'}\delta_{\bar{j}\bar{j}'}\delta_{\alpha\alpha'}\delta_{\beta\beta'} \\ \sum_{\substack{J \\ \alpha,\beta=1,\cdots Z_{j\bar{j}}}} \Psi_x^{(J;\alpha,\beta)}\Psi_{x'}^{(J;\alpha,\beta)\,*} &= \delta_{xx'}\ . \end{aligned} \tag{7.9}$$

Following a standard argument, in (7.8) the $\Psi^{(J;\alpha,\beta)}$ appear as the eigenvectors and the ratios $S_{ij}S_{\bar{i}\bar{j}}/S_{1j}S_{1\bar{j}}$ as the eigenvalues of the $\tilde{V}_{i\bar{i}}$ matrices. As the latter satisfy the double fusion algebra (7.5), so do the matrices $\tilde{V}$.

In fact the integer numbers $\tilde{V}_{ij;\,x}{}^y$ may be regarded not only as the entries of $|\widetilde{\mathcal{V}}| \times |\widetilde{\mathcal{V}}|$ matrices $\tilde{V}_{ij}$, $i, j \in \mathcal{I}$, as we just did, but also as those of $|\mathcal{I}| \times |\mathcal{I}|$ matrices $\tilde{V}_x{}^y$, $x, y \in \widetilde{\mathcal{V}}$.

By convention, the label 1 refers to the trivial (neutral) twist, and it is thus natural to impose the further constraint that for $y = z = 1$, $\tilde{V}$ reduces to the modular invariant matrix, up to a conjugation

$$\tilde{V}_{i\bar{i}^*;\,1}{}^1 = Z_{i\bar{i}}\ . \tag{7.10}$$

This is consistent with (7.8) if

$$\Psi_1^{(J;\alpha,\alpha')} = \sqrt{S_{1j}S_{1\bar{j}}}\,\delta_{\alpha\alpha'} =: \Psi_1^{(J)}\,\delta_{\alpha\alpha'}\ . \tag{7.11}$$

In particular $\Psi_1^{(1)} = S_{11}$ and denoting $\tilde{d}_x = \frac{\Psi_x^{(1)}}{\Psi_1^{(1)}}$, this implies, using the unitarity of Ψ and of the modular matrix S, the "completeness" relation

$$\sum_{x\in\tilde{\mathcal{V}}} \tilde{d}_x^2 = \frac{1}{(S_{11})^2} = \sum_{i\in\mathcal{I}} d_i^2\,, \tag{7.12}$$

(see also [10]). It also follows from (7.8) that $\tilde V_{ij;\,x}{}^{y} = \tilde V_{i^* j^*;\,y}{}^{x}$ and conversely, this latter property, together with (7.5), suffices to guarantee the diagonalisability of the $\tilde V$ in an orthonormal basis, as in (7.8).

Then, if we define the matrices $\mathbf{T}^x_{i,j} := \tilde V_{ij^*;\,1}{}^{x}$, thus $\mathbf{T}^1 = Z$, taking the $x = z = 1$ matrix element of (7.5) yields

$$\sum_x \mathbf{T}^x_{i,j}\ \mathbf{T}^x_{i'^*,j'^*} = \sum_{i'',j''} N_{ii'}{}^{i''}\, N_{jj'}{}^{j''}\, Z_{i''j''}\,, \tag{7.13}$$

which is the way the matrices $\mathbf{T}^x$ appeared originally in the work of Ocneanu, under the name of "modular splitting method".

The set of matrices $\tilde V_{ij}$ may be regarded as the adjacency matrices of a set of graphs with a common set of vertices $\widetilde{\mathcal{V}}$. In any RCFT, the fusion ring is generated by a finite number of representations f of $\mathcal{I}$ called fundamental, and because of (7.5), it is sufficient to give the graphs of $\tilde V_{f1}$ and of $\tilde V_{1f}$ for these representations to generate the whole set by fusion. (For example, for the $\widehat{sl}(2)$ theories, $\tilde V_{21}$ and $\tilde V_{12}$ suffice.) Following Ocneanu [1], it is convenient to represent these graphs simultaneously on the same chart, with edges of different colours. We shall refer to this multiple graph as the Ocneanu graph $\widetilde G$ associated with the graph G of the original theory. Examples are given in Fig. 10 of Appendix B for $\widehat{sl}(2)$ theories, and additional ones may be found in [2,11]. If one attaches the matrix $\mathbf{T}^x$ to vertex x, the two kinds of edges of the graph describe the action of the fusion matrices on the left and right indices of the $\mathbf{T}^x$. For example the edges of the first colour (red full lines on Fig. 10) encode the $\tilde V_{f1;\,y}{}^{z}$ in

$$N_{fi}{}^{i'}\,\mathbf{T}_{i'j}{}^{z} = \sum_y \tilde V_{f1;\,y}{}^{z}\ \mathbf{T}_{ij}{}^{y} \tag{7.14}$$

or in short, $(N_f \otimes I)\mathbf{T}^z = \sum_y \tilde V_{f1;\,y}{}^{z}\,\mathbf{T}^y$, and likewise, those of the second colour (blue, broken) describe $(I \otimes N_f)\mathbf{T}^z = \sum_y \mathbf{T}^y\,\tilde V_{1f;\,y}{}^{z}$.

7.2. The $\widetilde N$ algebra

In turn, this Ocneanu graph $\widetilde G$ may be used to define a new algebra in the same way as $\hat N$ was attached to the graph G. To each vertex x of the graph we attach a matrix $\widetilde N_x = \{\widetilde N_{yx}{}^{z}\}$ of size $|\widetilde{\mathcal{V}}| \times |\widetilde{\mathcal{V}}|$. For the special vertex 1, $\widetilde N_1 = I$. The matrices $\widetilde N$ are assumed to satisfy the algebra (1.5): $\tilde V_{ij}\,\widetilde N_x = \sum_z \tilde V_{ij;\,x}{}^{z}\,\widetilde N_z$, (compare with (7.2)). Using

the spectral decomposition (7.8) of the $\tilde{V}$, one may construct an explicit solution for these matrices $\widetilde{N}_x$

$$\widetilde{N}_{yx}{}^z = \sum_{J;\alpha} \sum_{\beta,\gamma} \Psi_y^{(J;\,\alpha,\beta)} \frac{\Psi_x^{(J;\,\beta,\gamma)}}{\Psi_1^{(J)}} \Psi_z^{(J;\,\alpha,\gamma)\,*} . \tag{7.15}$$

Taking into account the orthonormality of the Ψ, one finds that $\widetilde{N}_{1x}{}^z = \widetilde{N}_{x1}{}^z = \delta_{xz}$, and that $\widetilde{N}_x$ form a matrix representation of an algebra

$$\hat{E}_x\ \hat{E}_y = \sum_z \widetilde{N}_{xy}{}^z \hat{E}_z\ , \tag{7.16}$$

with an identity and a finite basis. The algebra is associative, but in general non-commutative if some $Z_{ij} > 1$. Indeed, if all $Z_{ij} = 1$, the summation over α, β, γ in (7.15) is trivial, and this equation is, once again, nothing else than the spectral decomposition of the matrices $\widetilde{N}$ in terms of the one-dimensional representations Ψ_x/Ψ_1 of the algebra. If, however, some $Z_{ij} > 1$, the matrices $\widetilde{N}$ are not simultaneously diagonalisable, but rather *block-diagonalisable* with blocks $\gamma_x^{(J;\beta,\gamma)} = \Psi_x^{(J;\beta,\gamma)}/\Psi_1^{(J)}$ forming a $Z_{j\bar{j}}$-dimensional representation of the algebra

$$\sum_\beta \gamma_x^{(J;\alpha,\beta)}\, \gamma_y^{(J;\beta,\gamma)} = \sum_z \widetilde{N}_{xy}{}^z\, \gamma_z^{(J;\alpha,\gamma)}\ . \tag{7.17}$$

(see also [10] (Lemma 5.2) for a similar although somewhat less explicit variant of (7.15), with $\widetilde{N}_{xy}^z = \langle \beta_z, \beta_x \circ \beta_y, \rangle$ being the "sector product matrices"). By inspection, one checks, at least in all ADE cases (see below and Appendix B), that these $\widetilde{N}$ matrices have non negative integer entries. They may indeed be viewed as multiplicities (of dual triangles with three white vertices), and accordingly the algebra (7.16) appears as the algebra of the center of $\hat{\mathcal{A}}$, with the product in the l.h.s. of (7.16) given by the vertical product of [1], compare with (3.23) and see Appendix A. These matrices are recovered also directly from (7.6), (7.7),

$$\widetilde{N}_{yx}{}^z = \mathrm{Tr}(X_y\ X_x\ X_z^\dagger) \tag{7.18}$$

using that $\mathrm{Tr}P^{(J;\,\alpha,\beta)} := \delta_{\alpha\beta}\, S_{1j}\, S_{1\bar{j}}$ (this definition of the trace may be justified in unitary CFT's in exactly the same way as the norm of the Ishibashi states, via the $\tau \to \infty$ asymptotics of the characters $\chi_j(\tau)$, see [61,3] and (4.2) of [8]). Equivalently, we have

$$X_x X_y = \widetilde{N}_{xy}{}^z X_z\ , \tag{7.19}$$

thus justifying the name of twist fusion algebra that we give to the $\widetilde{N}$ algebra. In this latter context, the non-commutativity of this $\widetilde{N}$ algebra may be viewed as coming from the inpenetrability and the resulting lack of commutativity of the defect lines to which the twists X_x and X_y are attached.

If a conjugation in the set $\widetilde{\mathcal{V}}$ is defined through

$$\Psi_{x^*}^{(J;\,\alpha,\beta)} = \big(\Psi_x^{(J;\,\beta,\alpha)}\big)^* \tag{7.20}$$

(note the reversal of the indices α and $\beta\,$!), it follows that

$$\widetilde{N}_{yx}{}^{1} = \delta_{xy^*}\ , \tag{7.21}$$

and the noncommutativity of $\widetilde{N}$ modifies the analogue of the symmetry relations (2.4) according to

$$\widetilde{N}_{yx}{}^{z} = \widetilde{N}_{x^* y^*}{}^{z^*} = \widetilde{N}_{zx^*}{}^{y}\,. \tag{7.22}$$

Equation (7.15) may be rewritten as a sum of $\sum_{j\bar{j};\alpha} 1 = \sum_{j\bar{j}} Z_{j\bar{j}}$ (matrix) idempotents $\mathbf{e}^{J;\alpha}_{yz;\beta,\gamma} = \frac{1}{Z_{j\bar{j}}}\,\Psi_y^{(J;\alpha,\beta)}\,\Psi_z^{(J;\alpha,\gamma)\,*}$,

$$(\mathbf{e}^{J;\alpha}\,\mathbf{e}^{J;\alpha})_{xz;\beta,\gamma} = \sum_{y,\gamma'} \mathbf{e}^{J;\alpha}_{xy;\beta,\gamma'}\ \mathbf{e}^{J;\alpha}_{yz;\gamma',\beta} = \mathbf{e}^{J;\alpha}_{xz;\alpha,\beta}\,, \qquad \sum_{J,\alpha,\beta} Z_{j\bar{j}}\,\mathbf{e}^{J;\alpha}_{\beta,\beta} = \widetilde{N}_1\,, \tag{7.23}$$

$$\tilde{N}_x = \sum_{J,\alpha}\sum_{\beta,\gamma} Z_{j\bar{j}}\,\gamma_x^{J;\beta,\gamma}\,\mathbf{e}^{J;\alpha}_{\beta,\gamma} = \sum_J Z_{j\bar{j}}\,\gamma_x^J \sum_\alpha \mathbf{e}^{J;\alpha}\,, \tag{7.24}$$

where, suppressing the matrix indices, the sum runs over the physical spectrum $(J,\alpha) := (J,\alpha,\alpha)$. These are the labels of the representations of the $\tilde{N}$ algebra, which are $Z_{j\bar{j}}$ -dimensional and given according to (7.17) by the matrices γ_x^J, i.e.,

$$\begin{aligned} &\triangle_{J;\alpha,\beta} : \tilde{N}_x \to \triangle_{J;\alpha,\beta}(\tilde{N}_x) = \gamma_x^{J;\alpha,\beta}\,,\\ &\triangle_{J;\alpha,\gamma}(\tilde{N}_x\,\tilde{N}_y) = \sum_\beta \triangle_{J;\alpha,\beta}(\tilde{N}_x)\triangle_{J;\beta,\gamma}(\tilde{N}_y) = \sum_z \tilde{N}_{xy}^z\,\triangle_{J;\alpha,\gamma}(\tilde{N}_z)\,. \end{aligned} \tag{7.25}$$

The formula (7.24) is then interpreted as a decomposition of the regular representation of the $\widetilde{N}$-algebra into a sum of representations $\triangle_J$ each appearing with multiplicity $Z_{j\bar{j}}$ so the dimension is $\sum_{j\bar{j}} Z_{j\bar{j}}\ \dim(\triangle_J)$=$\sum_{j\bar{j}} Z^2_{j\bar{j}} = |\widetilde{\mathcal{V}}|$. In [11] a formula analogous to (7.25), or (7.17) appears directly for the elements $\hat{E}_x$ in $\hat{\mathcal{A}}$ spanning (with respect to the vertical product) the algebra (1.4), see Appendix A.

We now return to the graph algebra $\hat{N}$ of the chiral graph G mentioned in the introduction to this section. In fact, we shall restrict our attention to type I cases, which are the only ones for which all the matrix elements of the $\hat{N}_a$ are non negative integers. Because in this case, equation (7.3) applies, each exponent appears $(n_{j1}{}^a)^2$ times for each representation a of the extended algebra, identified with a vertex $a \in T$. It is advantageous to denote the corresponding eigenvectors of the n matrices as $\psi^{(j,a;\alpha,\beta)}$, with $\alpha, \beta = 1, \cdots, n_{j1}{}^a$. In [18], various formulae have been established for the components $\psi_b^{(j,a)}$, $b \in T$. It is easy to extend them to

$$\begin{aligned} \psi_1^{(j,a;\alpha,\beta)} &= \delta_{\alpha\beta}\, \psi_1^{(j,a)}\,, \quad \psi_1^{(j,a)} = \sqrt{S_{1j} S^{\text{ext}}_{1a}}\,, \quad \text{for } n^a_{j1} \neq 0\,, \\ \psi_b^{(j,a;\alpha,\beta)} &= \delta_{\alpha\beta}\, \psi_1^{(j,a)}\ \frac{S^{\text{ext}}_{ba}}{S^{\text{ext}}_{1a}}\,, \qquad \text{for } a, b \in T\ , \end{aligned} \tag{7.26}$$

using the modular invariance identity

$$\sum_{i \in \mathcal{I}} S_{ij}\ n_{i1}{}^b = \sum_{a \in T} S^{\text{ext}}_{ba}\ n_{j1}{}^a\,, \qquad b \in T\,. \tag{7.27}$$

The similarity with the case of the $\widetilde{N}$ algebra (of which the $\hat{N}$ algebra turns out to be a subalgebra in these type I cases) suggests a formula which encompasses and generalises all known cases

$$\hat{N}_{cb}{}^d = \sum_{\substack{a \in T\,, j \in \mathcal{I} \\ \alpha,\beta,\gamma = 1,\cdots,n_{j1}{}^a}} \psi_c^{(j,a;\,\alpha,\beta)}\ \frac{\psi_b^{(j,a;\,\beta,\gamma)}}{\psi_1^{(j,a)}}\ \psi_d^{(j,a;\,\alpha,\gamma)\,*}\,. \tag{7.28}$$

It is an easy matter to check that the relations (7.1) and (7.2) are indeed satisfied. We have checked in the simplest case $n = 2$ of $\widehat{sl}(2n)_{2n} \subset \widehat{so}(4n^2-1)_1$, for which multiplicities $n_{i1}{}^a > 1$ are known to occur, and we conjecture in general, that this formula always yields non negative integers, and gives an explicit realisation of the considerations of [49,62] [6].

7.3. The $\tilde{n}$ matrices

We then introduce a new set of matrices $\tilde{n}_x = \{\tilde{n}_{ax}{}^b\}$, $a, b \in \mathcal{V}$, which form a non negative integer valued representation (nimrep) of this $\widetilde{N}$ algebra, see (1.3), in clear analogy with

[6] It is understood that in cases where exponents come with a non trivial multiplicity, the remaining arbitrariness in the choice of the ψ is used to make the $\hat{N}$ nonnegative integers, and this seems always possible in type I cases.

(1.1). [7] Like the $\widetilde{N}$, these matrices are non-commuting in general, if some $Z_{j\bar{j}} > 1$, and they admit a block decomposition like (7.15)

$$\tilde{n}_{ax}{}^{b} = \sum_{j} \sum_{\alpha,\beta=1,\cdots,Z_{jj}} \psi_a^{j,\alpha} \, \frac{\Psi_x^{(j,j;\,\alpha,\beta)}}{\Psi_1^{(j,j)}} \, \psi_b^{j,\beta\,*} = \tilde{n}_{bx^*}{}^{a} \,. \tag{7.29}$$

One also checks, using the orthonormality and conjugation properties of the ψ and Ψ, that

$$\sum_{x\in\widetilde{\mathcal{V}}} \tilde{n}_{ax}{}^{a'} \tilde{n}_{b'x^*}{}^{b} = \sum_{i\in\mathcal{I}} n_{ia}{}^{b} n_{i^*b'}{}^{a'} \ . \tag{7.30}$$

These matrices are again interpreted as multiplicities: namely $\tilde{n}_{ax}{}^{b}$ describes the dimension of the space $\hat{V}_{ax}^{b}$ of dual triangles with fixed markings x, a, b (one black, two white vertices). Varying a, b, they form a basis of the dual vector space $\hat{V}_x$. Then (1.3) serves as a consistency condition needed to give sense to the dual (vertical) product $\hat{V}_x \otimes_v \hat{V}_y$, in which $\hat{V}_z$ appears with multiplicity $\widetilde{N}_{xy}{}^{z}$, the latter replacing the Verlinde multiplicities in a formula analogous to (3.3). On the other hand (7.30) is interpreted as the equality between the dimensions of the space of double triangles and that of dual double triangles with a, a', b, b' fixed and justifies a change of basis considered in Appendix A (see Fig. 9) Recalling that in section 3, $m_j = \sum_{a,b\in\mathcal{V}} n_{ja}{}^{b}$ stands for the dimension of the space V^j of triangles (or CVOs), we now denote $\widetilde{m}_x = \sum_{a,b\in\mathcal{V}} \tilde{n}_{xa}{}^{b}$ the dimension of the space $\hat{V}_x$. The equality of the dimensions of the double triangle algebra $\mathcal{A}$ and of its dual $\hat{\mathcal{A}}$ amounts to the identity

$$\sum_{j\in\mathcal{I}} m_j^2 = \sum_{x\in\widetilde{\mathcal{V}}} \widetilde{m}_x^2 \tag{7.31}$$

which results indeed from the summation over a, a', b, b' in (7.30). On the other hand a less trivial equality holds, checked case by case in all $sl(2)$ cases,

$$\sum_{j\in\mathcal{I}} m_j = \sum_{C_x\ ,\, x\in\widetilde{\mathcal{V}}} \tilde{m}_x \tag{7.32}$$

where the sum in the r.h.s. runs over the "classes" C_x in $\widetilde{\mathcal{V}}$ (or classes in the $\widetilde{N}$ algebra), determined by $x \sim y$, iff $\forall J,\ \mathrm{tr}(\triangle_J(\widetilde{N}_x)) = \sum_{\alpha,\alpha} \gamma_x^{J,\alpha,\alpha} = \mathrm{tr}(\triangle_J(\widetilde{N}_y))$, i.e., the characters

[7] In the subfactor approach, given an inclusion of subfactors $N \subset M$, the equality (1.3) is interpreted as an associativity condition for the $M-M$, $M-N$ sectors, similarly as the analogous identity (1.1) for the $N-N$, $N-M$ sectors [11].

of the representations of the $\widetilde{N}$ algebra are constant on the class C_x. For cases with trivial multiplicities $Z_{j\bar{j}} = 0, 1$ the summation in the r.h.s. runs over the set $\widetilde{\mathcal{V}}$ and (7.32) expresses the equality of dimensions of the regular representations of $\mathcal{A}$ and $\hat{\mathcal{A}}$. In the $sl(2)$ $D_{\rm even}$ cases there are two nontrivial classes C_x, each containing the fork vertices in the chiral subgraphs of $\widetilde{G}$, see Appendix B.

The physical interpretation of the matrices (7.29) is obtained by looking at the effect of a twist in the presence of boundaries. One consider the RCFT on a finite cylinder with boundary states $|a\rangle$ and $\langle b|$ at the ends and a twist operator $X_x^\dagger$ in between. Repeating the calculations of partition functions carried out in [3,8], one finds that the "open string channel" is described by a Hilbert space with representation $\mathcal{V}_i$ occurring with multiplicity $(n_i \tilde{n}_x)_{a^*}{}^{b^*}$, i.e. the matrix element of a (commuting) product of the matrices n_i and $\tilde{n}_x$. Thus $(\tilde{n}_x)_{a^*}{}^{b^*}$ is the multiplicity of the identity character in this "twisted" cylinder partition function.

7.4. The $\widetilde{M}$ matrices

The last set of matrices that we may associate with the Ocneanu graph generalises the Pasquier algebra. We can define a dual (in the sense of [58]) of the $\widetilde{N}$ algebra by the algebra of linear maps

$$\begin{aligned}
\triangle^+_{J;\beta,\beta'}:\ \widetilde{N}_x \to \triangle^+_{J;\beta,\beta'}(\widetilde{N}_x) &= \frac{\Psi_1^{(J;\beta,\beta')}}{\Psi_x^1}\,\triangle_{J;\beta,\beta'}(\widetilde{N}_x) = \frac{\Psi_x^{(J;\beta,\beta')}}{\Psi_x^1}\ , \\
(\triangle^+_{I;\alpha,\alpha'}\,\triangle^+_{J;\beta,\beta'})(\widetilde{N}_x) &= \triangle^+_{I;\alpha,\alpha'}(\widetilde{N}_x)\,\triangle^+_{J;\beta,\beta'}(\widetilde{N}_x) \\
&= \sum_{K;\gamma,\gamma'} \widetilde{M}_{(I;\alpha,\alpha')\,(J;\beta,\beta')}{}^{(K;\gamma,\gamma')}\ \triangle^+_{K;\gamma,\gamma'}(\widetilde{N}_x)
\end{aligned} \tag{7.33}$$

with structure constants

$$\widetilde{M}_{(I;\alpha,\alpha')\,(J;\beta,\beta')}{}^{(K;\gamma,\gamma')} = \sum_x \frac{\Psi_x^{(I;\alpha,\alpha')}}{\Psi_x^{(1)}}\,\Psi_x^{(J;\beta,\beta')}\,\Psi_x^{(K;\gamma,\gamma')\,*}\,. \tag{7.34}$$

This algebra is abelian and its 1-dimensional representations, or characters, are given by (7.33). An involution $(*)$ in the set $\{(I;\alpha,\alpha')\}$ is defined by the complex conjugation $\Psi_x^{(I;\alpha,\alpha')^*} = \Psi_x^{(I;\alpha,\alpha')\,*}$ so that $M_{(I;\alpha,\alpha')^*} = {}^tM_{(I;\alpha,\alpha')}$. The subset of the numbers formed by the $\widetilde{M}_{(I;\,\alpha,\alpha)(J;\,\beta,\beta)}{}^{(K;\,\gamma,\gamma)}$, i.e. diagonal in the multiplicity indices, plays a physical rôle. Their explicit computation (in the ADE cases) shows that (i) they are non negative

algebraic numbers; (ii) they give the modulus squares of the relative structure constants of the OPA of the corresponding CFT

$$|d_{(I;\,\alpha)(J;\,\beta)}{}^{(K;\,\gamma)}|^2 = \widetilde{M}_{(I;\,\alpha,\alpha)\,(J;\,\beta,\beta)}{}^{(K;\,\gamma,\gamma)}\ . \tag{7.35}$$

We recalled in section 5 that the Pasquier algebra gives access to the relative structure constants of spinless fields. The OPA structure constants of non left-right symmetric fields, however, were escaping in general this determination in terms of graph-related data [8]. The empirical result in [17] only states that in the cases of conformal embeddings D_4, E_6, E_8 the l.h.s. of (7.35) factorises into a product of scalar constants (and hence is expressed by the Pasquier algebra structure constants) and that for the $D_{\rm even}$ series this factorisation holds in a somewhat weaker sense; this factorisation is confirmed (see Appendix B) by what is computed for the r.h.s.

In fact (7.35) can be derived extending the consideration of [8] to 4-point functions of physical fields in the presence of twists; it is sufficient to look at the functions on the plane, which can be interpreted as the $L/T \to \infty$ limit of the torus correlators, $\lim_{L/T\to\infty}$ Tr$(e^{-2LH}$ ......), when we map it to the plane through $w \to z = {}^{\frac{-2\pi i w}{T}}$. Let us sketch the argument which is a generalisation of the derivation of the locality equations; we shall use the convention of notation in [18]. We consider a 4-point function with insertion of two twist operators (7.6) (omitting the labels (P) on the fields and the OPE coefficients)

$$\begin{aligned}&\langle 0|\Phi_{(J^*;\beta^*)}(z_1,\bar z_1)\,\Phi_{(I^*;\alpha^*)}(z_2,\bar z_2)\,X_x\,\Phi_{(I;\alpha')}(z_3,\bar z_3)\,\Phi_{(J;\beta')}(z_4,\bar z_4)\,X_x^\dagger|0\rangle\\ &\quad=\sum_{k,\bar k,\gamma,\gamma'} d^{(1)}_{(J^*;\beta^*)(J;\beta)}\, d^{(J;\beta)}_{(I^*;\alpha^*)(K;\gamma,\gamma')}\,\frac{\Psi_x^{(k,\bar k;\gamma,\gamma')}}{\Psi_1^{(k,\bar k)}}\, d^{(K;\gamma,\gamma')}_{(I;\alpha')(J;\beta')}\,\frac{\Psi_x^{(1)}}{\Psi_1^{(1)}}\\ &\qquad\langle 0|\phi^1_{j^*j}(z_1)\,\phi^j_{i^*k}(z_2)\,\phi^k_{ij}(z_3)\,\phi^i_{i1}(z_4)|0\rangle\times(\text{right chiral block})\,,\end{aligned}\tag{7.36}$$

taking into account that $d^{(J;\beta')}_{(J;\beta')(1)} = 1$. The limit $z_{21}, z_{34} \to \infty$ of this correlator is alternatively represented by the identity contribution in the correlator

$$\begin{aligned}&\langle 0|\Phi_{(I^*;\alpha^*)}(z_2,\bar z_2)\ X_x\,\Phi_{(I;\alpha')}(z_3,\bar z_3)\,\Phi_{(J;\beta')}(z_4,\bar z_4)\,X_x^\dagger\,\Phi_{(J^*;\beta^*)}(z_1,\bar z_1)|0\rangle\\ &\quad=\sum_{p,\bar p,\delta,\delta'} d^{(1)}_{(I^*;\alpha^*)(I;\alpha)}\,\frac{\Psi_x^{(i,\bar i;\alpha,\alpha)}}{\Psi_1^{(i,\bar i)}}\, d^{(I;\alpha)}_{(I;\alpha')(P;\delta,\delta')}\, d^{(P;\delta,\delta')}_{(J;\beta')(J^*;\beta^*)}\,\frac{\Psi_x^{(j,\bar j;\beta,\beta)}}{\Psi_1^{(j,\bar j)}}\\ &\qquad\langle 0|\phi^1_{i^*i}(z_2)\,\phi^i_{ip}(z_3)\,\phi^p_{jj^*}(z_4)\,\phi^{j^*}_{j^*1}(z_1)|0\rangle\times(\text{right chiral block})\,,\end{aligned}\tag{7.37}$$

[8] The equation (5.7) represents the constants d in terms of the $3j$- and $6j$-symbols and the general nonscalar reflection coefficients.

i.e., by the first term $p=1=\bar{p}$. Next we use the braiding relations for the chiral blocks to identify the two products of chiral correlators , i.e., move j^* and $\bar{j}^*$ to the very right– this brings about the product of fusing matrices $F_{kp}\left[\begin{smallmatrix} j^* & j\\ i & i\end{smallmatrix}\right]\, F_{\bar{k}\bar{p}}\left[\begin{smallmatrix} \bar{j}^* & \bar{j}\\ \bar{i} & \bar{i}\end{smallmatrix}\right]$ taken at $p=1=\bar{p}$. This implies $\alpha=\alpha'$ and $\beta=\beta'$ and also trivialises the fusion matrices to the ones in the diagonal counterpart of (3.11), i.e., we get ratios of square roots of q-dimensions, which precisely match the factors Ψ_1 (7.26) coming from the twists. Equating the coefficients and taking also into account the symmetries of the OPE coefficients (this produces the same sign $(-1)^{s_I+s_J}$ in both sides), see [18], we finally obtain

$$\sum_{k,\bar{k},\gamma,\gamma'} |d^{(K;\gamma,\gamma')}_{(I;\alpha)(J;\beta)}|^2\ \frac{\Psi_x^{(K;\gamma,\gamma')}}{\Psi_x^{(1)}} = \frac{\Psi_x^{(I;\alpha,\alpha)}}{\Psi_x^{(1)}}\ \frac{\Psi_x^{(J;\beta,\beta)}}{\Psi_x^{(1)}}\,, \tag{7.38}$$

from which (7.35) follows. In deriving (7.38) we have assumed that the decomposition of the physical fields involves several copies of each product of left and right chiral blocks, i.e., $\Phi_{I;\alpha}(z,\bar{z}) = \sum_{j,\bar{j},k,\bar{k},\beta,\beta',\gamma,\gamma'}\ d^{(K;\gamma,\gamma')}_{(I;\alpha)(J;\beta,\beta')}\ \left(\phi^k_{ij}(z)\otimes\phi^{\bar{k}}_{\bar{i}\bar{j}}(\bar{z})\right)^{(\gamma,\gamma')}_{(\alpha,\alpha)(\beta,\beta')}$. These copies are labelled by the pairs (β,β') , (γ,γ') and they correspond to the multiplicity of states in the projectors $P_x^{(k,\bar{k};\gamma,\gamma')}$ in (7.6); in the only nontrivial $sl(2)$ case, the $D_{\rm even}$ series, $\widetilde{M}^{(K;\gamma,\gamma')}_{(I;\alpha,\alpha)(J;\beta,\beta)}$, and hence $d^{(K;\gamma,\gamma')}_{(I;\alpha)(J;\beta)}$, are identically zero for $\gamma\neq\gamma'$. In the previous discussion we have suppressed for simplicity the multiplicity indices $t=1,2,\dots,N_{ij}{}^k$ and $\bar{t}=1,2,\dots,N_{\bar{i}\bar{j}}{}^{\bar{k}}$ appearing in the higher rank cases; when restored the modulus square in the l.h.s. of (7.38) and (7.35) is replaced by $\sum_{t,\bar{t}}\,|d^{(K;\gamma,\gamma;\,t,\bar{t})}_{(I;\alpha)(J;\beta)}|^2$. Note that in the presence of a twist operator the identity 1-point function appears normalised as $\langle 0|\Phi_{(1)}\,X_x|0\rangle = \frac{\Psi_x^{(1)}}{\Psi_1^{(1)}} = \tilde{d}_x$.

An intriguing issue is the fact that from a mathematical point of view, the indices x play a rôle dual to that of representation labels $i\in\mathcal{I}$, (dual in the algebraic sense, going from a linear space to the space of its linear functionals, see Appendix A and also equation (7.12)), while from a physical point of view, they play a rôle dual to that of the labels of bulk fields: this is apparent in equation (7.15) where there is a (Fourier-like) duality between the set $\widetilde{\mathcal{V}}$ of x and the set $\widetilde{\rm Exp}$ of pairs (J) counted with a multiplicity $(Z_{j\bar{j}})^2$, that is between the vertices and the "exponents" of the graph $\widetilde{G}$.

We conclude this subsection with the remark that some correlators including twist operators may be interpreted as generalised order-disorder field correlators, compare with [38], where such functions matching the operator content of the $\mathbb{Z}_2$-twisted torus partition functions of [63,64] were constructed and their OPE coefficients computed. We recall [8] that the partition functions of [63,64] provide the simplest examples of solutions of (7.13).

7.5. Constructing the $\widetilde{G}$ graphs

Let us see now how the matrices $\tilde{V}_1{}^x$ and Ψ, from which the graph $\widetilde{G}$ and all the other matrices $\tilde{V}$, $\widetilde{N}$, $\tilde{n}$ and $\widetilde{M}$ may be constructed, can be determined in a given CFT, i.e. starting from a given modular invariant Z and the associated graph G. (see [65] for a detailed discussion of the particular E_6 case of $\widehat{sl}(2)$ following a different approach.)

First, in the case of a diagonal theory, $Z_{ij} = \delta_{ij}$, it is natural to identify the set $\widetilde{\mathcal{V}}$ with the set $\mathcal{I}$ of representations, since their cardinality agrees, and to take

$$\tilde{V}_{ij} = N_i \; N_j \tag{7.39}$$

understood as a matrix product, i.e. $\tilde{V}_{ij;\,x}{}^y = \sum_{k\in\mathcal{I}} N_{ix}{}^k N_{jk}{}^y$, in particular $\tilde{V}_{ij;\,1}{}^k = N_{ij}{}^k$. The corresponding $\Psi_x^{(j,j)}$ are just the modular matrix elements S_{xj} and the Ocneanu graph $\widetilde{G} = \widetilde{A}$, which is generated by the "fundamental" $\tilde{V}_{f1}$ and $\tilde{V}_{1f}$, both equal to N_f, is identical to the ordinary graph $G = A$.

As a second case, consider a non-diagonal theory with a matrix $Z_{ij} = \delta_{i\zeta(j)}$, where ζ is the conjugation of representations or some other automorphism of the fusion rules (like the Z_2 automorphism in the $D_{2\ell+1}$ cases of $\widehat{sl}(2)$ theories). Then $\widetilde{\mathcal{V}} = \mathcal{I}$, and $\tilde{V}_{ij} = \tilde{V}^{(\mathrm{diag})}_{i\zeta(j)} = N_i N_{\zeta(j)}$. The graph is generated by $\tilde{V}_{f1} = N_f$ and $\tilde{V}_{1f} = N_{\zeta(f)}$, each one giving a subgraph isomorphic to A. (see Fig. 10 for the case of D_{odd}). The $\Psi_x^{(J)} = S_{xj}\delta_{j\zeta(\bar{j})}$, and one finds that the $\widetilde{N}$ matrices reduce to those of the diagonal case, i.e. $\widetilde{N}_{xy}{}^z = N_{xy}{}^z$, $x, y, z \in \mathcal{I}$, while $\tilde{n}_{ax}{}^b = n_{xa}{}^b$ and $\widetilde{M}_{(i,\zeta(i))\,(j,\zeta(j))}{}^{(k,\zeta(k))} = N_{ij}{}^k$.

General expressions may be obtained for type I theories (7.3). The algebra (7.1) enables one to define a partition of the set $\mathcal{V}$ into equivalence classes T_κ: $a \sim a'$ if $\exists\, b \in T : \hat{N}_{ab}{}^{a'} \neq 0$ [58,14]. The number of such classes equals the number of representations of $\mathfrak{A}$ coupled to the identity in the modular invariant, i.e. of $i \in \mathcal{I}$ such that $Z_{1i} \neq 0$ [9]. Since (7.3) applies to the matrix $\mathbf{T}^1 = Z$, it suggests to look for similar expressions for the other $\mathbf{T}^x$. We find that in all known type I cases, in particular for $\widehat{sl}(2)$ theories, the labels x may be taken of the form (a, b, κ), $a, b \in \mathcal{V}$, κ a class label, and

$$\mathbf{T}^x_{ij} = \tilde{V}_{ij^*;\,1}{}^{(a,b,\kappa)} = P^{(\kappa)}_{ab} := \sum_{c\in T_\kappa} n_{ic}{}^a n_{jc}{}^b \tag{7.40}$$

[9] This may be established in cases where the $\hat{N}$ algebra is commutative, and where the structure constants of both $\hat{N}$ and its dual M are non negative, following the work of [58]. It seems to extend to non-commutative cases as well, as we checked on the aforementioned case of $\widehat{sl}(4)_4$, where some entries of M are negative or even imaginary.

with c running over a certain subset T_κ of vertices, or equivalently

$$\tilde{V}_{ij;\,1}{}^{(a,b,\kappa)} = \sum_{c\in T_\kappa} n_{ic}{}^a n_{jb}{}^c \ . \tag{7.41}$$

One checks that indeed $\mathbf{T}^1 = Z$, the modular invariant matrix as given in (7.3). As the matrices n_i form a representation of the fusion algebra, $n_i n_j = N_{ij}{}^k n_k$, one finds that upon left multiplication by any N_f,

$$N_f.P^{(\kappa)}_{ab} = \sum_{a'} n_{fa'}{}^a \ P^{(\kappa)}_{a'b} \tag{7.42}$$

and likewise, upon right multiplication $P^{(\kappa)}_{ab}.N_{f^*} = \sum_d P^{(\kappa)}_{ad} n_{fd}{}^b$ by repeated use of $n_f^T = n_{f^*}$.

In theories like $\mathbb{Z}_n$ orbifolds of $\widehat{sl}(n)$ theories, there is a partition of the set of vertices into classes T_α such that

$$\forall a, a' \in T_\alpha \ , \ \forall b \in T_\beta \neq T, \qquad \hat{N}_{ba}{}^{a'} = 0 \ , \tag{7.43}$$

because the $\hat{N}$ algebra respects the $\mathbb{Z}_n$ grading of the vertices. Then let us prove that (7.13) follows from the Ansatz (7.41) with $x = (a, 1, \kappa)$

$$\begin{aligned} \sum_a \sum_\kappa (P^{(\kappa)}_{a1})_{ij} (P^{(\kappa)}_{a1})_{i'^* j'^*} &= \sum_{c\sim c'} \sum_{i''\in\mathcal{I}} N_{ii'}{}^{i''} \sum_d n_{i''1}{}^d \, \hat{N}_{dc'}{}^c \, n_{j1}{}^c \, n_{j'1}{}^{c'^*} \\ &= \sum_{c,c'} \sum_{i''\in\mathcal{I}} N_{ii'}{}^{i''} \sum_{d\in T} n_{i''1}{}^d \, \hat{N}_{dc'}{}^c \, n_{j1}{}^c \, n_{j'1}{}^{c'^*} \\ &= \sum_{i'',j''} N_{ii'}{}^{i''} \, N_{jj'}{}^{j''} Z_{i''j''} \end{aligned} \tag{7.44}$$

where we have repeatedly used (7.2) and (1.1) and on the second line, we have used (7.43) to restrict the summation over d to the set T; the constraint $c \sim c'$ is then automatically enforced, which enables us to sum over independent c and c'.

For the case of a conformal embedding $\hat{\mathfrak{h}}_k \subset \hat{\mathfrak{g}}_1$, we checked in all $\widehat{sl}(2)$ cases and conjecture in general that the label κ may be dropped, and x represented by a pair of vertices (a, b), $a \in \mathcal{V}$, and b running over a subset of vertices. Then we make use of formula (7.26) to express the eigenvectors $\psi_c^{j,d;\,\alpha,\beta}$, $c \in T$ in terms of the modular $S^{\rm ext}$

matrix of the extended algebra (i.e. of the $\hat{\mathfrak{g}}_1$ current algebra). In that way we find, multiplying (7.41) with $S_{i^*j}S_{\bar{i}^*\bar{j}}$,

$$\sum_{\gamma} \Psi_x^{(J;\gamma,\gamma)} = \sum_{d,\alpha,\bar{\alpha}} \frac{\psi_a^{(j,d;\,\alpha,\alpha)}\ \psi_b^{(\bar{j},d;\,\bar{\alpha},\bar{\alpha})\,*}}{S_{1d}^{\rm ext}}\,, \qquad x=(a,b) \tag{7.45}$$

where the sum in the l.h.s. runs according to $\gamma = 1,\cdots, Z_{j\bar{j}} = \sum_{d\in T} n_{j1}{}^d\, n_{\bar{j}1}{}^d$, and that in the r.h.s. runs on $\alpha = 1,\cdots, n_{j1}{}^d$, $\bar{\alpha} = 1,\cdots, n_{\bar{j}1}{}^d$. If there is only one $d\in T$ in the sum we can identify $\gamma = (\alpha,\bar{\alpha})$, if there are more, first γ has to be split into a multiple index and then each identified with a pair $(\alpha,\bar{\alpha})$ depending on d. For $a\in T$ $\Psi_a^{(J;\gamma,\gamma')} = \delta_{\gamma\gamma'}\,\Psi_1^{(J)}\,\frac{S_{ad}^{\rm ext}}{S_{1d}^{\rm ext}}$ is consistent with (7.45) and implies that $\widetilde{N}_{ab}{}^c = {}^{\rm ext}N_{ab}{}^c$ for $a,b,c\in T$, using $\sum_{j,\bar{j}\in d} Z_{j\bar{j}}\, S_{1j}\, S_{1\bar{j}} = (S_{1d}^{\rm ext})^2$, and hence, that $\widetilde{N}_a$, $a\in T$ form a subalgebra isomorphic to the extended fusion algebra.

In particular in the cases with commutative $\widetilde{N}$ algebra one computes

$$\begin{aligned}
\tilde{V}_{ij;\,(a_1,b_1)}{}^{(a_2,b_2)} &= \sum_{c\in T} (n_i\,\hat{N}_{a_1})_c{}^{a_2}\,(n_{j^*}\,\hat{N}_{b_1})_c{}^{b_2}\,,\\
\tilde{N}_{(a_1,b_1)\,(a_2,b_2)}{}^{(a_3,b_3)} &= \sum_{c\in T} (\hat{N}_{a_1}\,\hat{N}_{a_2})_c{}^{a_3}\,(\hat{N}_{b_1}\,\hat{N}_{b_2})_c{}^{b_3}\\
\tilde{n}_{(a,b)} &= \hat{N}_a\,\hat{N}_b\,,
\end{aligned} \tag{7.46}$$

which ensures that these matrices are integral, nonnegative valued. In particular

$$\tilde{V}_{i1;\,(a_1 1)}{}^{(a_2,1)} = n_{ia_1}{}^{a_2}\,, \quad \tilde{V}_{1j;\,(1,b_1)}{}^{(1,b_2)} = n_{jb_1}{}^{b_2}\ .$$

The description of $\mathbf{T}^x$ as a bilinear form in the n matrices does not seem to restrict to type I theories like in (7.41). Indeed, this is what happens in the $D_{\rm odd}$ and E_7 cases of $\widehat{sl}(2)$ theories. Knowledge of the $\mathbf{T}^x$ matrices (and in general of $\tilde{V}^x_{ij;(1,1,\kappa)}$ for any κ) determines the whole structure. It is easy to invert the (block-)diagonalisation formula of the $\mathbf{T}^x$ and to get, using also (7.11),

$$\sum_{\gamma} \Psi_x^{(J;\gamma,\gamma)\,*} = \sqrt{S_{1j}S_{1\bar{j}}}\,\sum_{i,\bar{i}\in\mathcal{I}} \mathbf{T}^x_{i\bar{i}^*} S_{i^*j} S_{\bar{i}^*\bar{j}}\ , \tag{7.47}$$

This determines Ψ_x completely for cases with $Z_{j\bar{j}}=1$, while higher multiplicities $Z_{j\bar{j}}>1$ require a little more work and care, see Appendix B for an illustration on the $D_{2\ell}$ case of $\widehat{sl}(2)$. Once Ψ is known, it is a simple matter to obtain all $\tilde{n}, \widetilde{N}, \tilde{V}$ and $\widetilde{M}$ matrices.

8. Conclusions and perspectives

The reader who has followed us that far should by now be convinced of the relevance and utility of Ocneanu's DTA $\mathcal{A}$ in the detailed study of rational CFT. In our view, two new concepts developed in this paper in connection with this quantum algebra have proved particularly useful:

- the generalised CVO, which are covariant under the action of $\mathcal{A}$, unify the treatment of bulk and boundary fields and permit a more direct discussion of their operator algebras;
- the twist operators, whose rôle manifests itself in several ways, give a physical interpretation to the abstract labels x of the dual algebra $\hat{\mathcal{A}}$ and to the coefficients $\widetilde{N}_{xy}{}^{z}$ and also, through their interplay with bulk fields, provide a new way to determine the general OPA structure constants in the bulk.

Several points deserve further investigation. First, as already pointed out in the Introduction, many of our statements which rely on the explicit examination of particular cases, mainly based on $sl(2)$ and $sl(3)$, and which are presented as conjectures in general, should be extended in a systematic way to all RCFT. The case of orbifold theories, in which the relevant graphs would be *affine* Dynkin diagrams and their generalisations, should be quite instructive. Other directions of generalisations include irrational CFTs (generic $c \geq 1$ CFTs or $N = 2$ superconformal CFTs) or non compact theories like Liouville [66].

Secondly, among the five types of 3-chains ${}^*F$ attached to the tetrahedra of Fig. 1, only two, namely F and ${}^{(1)}F$ have received a physical interpretation, as they underlie both the CFT and the related integrable critical lattice models. Understanding the meaning of the others, which all involve one or several twist labels x, presumably requires a deeper discussion of the interplay of twist operators with bulk and/or boundary fields.

In fact the general properties of twists and their relations with "twisted" representations of the underlying chiral algebra $\mathfrak{A}$ await a good discussion. We regard as quite significant that *all* partition functions either on a torus or on a cylinder with or without defect lines (twists) are expressible as linear or bilinear forms with non-negative integer coefficients of the $|\mathcal{V}|$ linear combinations of characters

$$\hat{\chi}_a := \sum n_{i1}{}^{a} \chi_i = Z_{a|1} \,. \tag{8.1}$$

This follows from eqs (7.3),(7.4) and from our Ansatz (7.41) that in type I the matrices $\mathbf{T}^x$ are bilinear in the n's. (In type II theories, we recall that the n's that appear here

are those of the parent type I theory). The $\hat{\chi}_a$ thus appear as the building blocks of all partition functions. Their natural interpretation, as alluded above, is that they are the characters of a class of more general representations of the extended algebra $\mathfrak{A}^{\rm ext}$. Among them, the subset $a \in T$ represents the ordinary, untwisted, representations. The other have been called twisted [3], or solitonic [55]. The induction/restriction method [49,62] of constructing these "twisted sectors" essentially amounts to the recursive solution of the system of equations (1.1), (7.2), (7.1). On the other hand, the direct definition of (some) of these twisted representations, closer in spirit to the concept of twist as developed in section 7, has been achieved only in a limited number of cases, see e.g. [56].

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Appendix A. Ocneanu DTA – dual structure

This appendix contains some more details on the Ocneanu DTA [1,11] and its WHA interpretation [4].

The coproduct (3.18) does not preserve the identity, i.e.,

$$\triangle(1_v) = \sum_b \sum_{i,j,a,c,\alpha,\gamma} e^{(cb)(cb)}_{i\,,\alpha,\alpha} \otimes e^{(ba)(ba)}_{j\,,\gamma,\gamma} \Big(=: 1_{v\,(1)} \otimes 1_{v\,(2)} \Big) \neq 1_v \otimes 1_v \tag{A.1}$$

while $\triangle(1) = 1 \otimes 1$ is one of the axioms of a Hopf algebra.

For $u\,, w \in \mathcal{A}$ and uw – the matrix (vertical) product, one has

$$\varepsilon(u\,w) = \sum_{i,j,a,b,c,\alpha,\gamma} \varepsilon(u\, e^{(cb)(cb)}_{i\,,\alpha,\alpha})\ \varepsilon(e^{(ba)(ba)}_{j\,,\gamma,\gamma}\ w) \Big(=: \varepsilon(u\, 1_{v\,(1)})\ \varepsilon(1_{v\,(2)}\, w) \Big)\,, \tag{A.2}$$

e.g. for $u = \sum_{a,b} C_{a,b} e_1^{aa,bb}$, $w = \sum_{a,b} C'_{a,b} e_1^{aa,bb}$ – one gets $\varepsilon(u\,w) = \mathrm{tr}(CC')$, and $\varepsilon(u)\,\varepsilon(w) = \sum_{a,b} C_{ab} \sum_{a',b'} C'_{a'b'} \neq \varepsilon(u\,w)$ in general, while the counit of a Hopf algebra is an algebra homomorphism.

The antipode is a linear anti-homomorphism $S(uw) = S(w)\,S(u)$, defined according to (3.21), and so that $S^{-1}(u) = (S(u^*))^*$. It is also an anti-cohomomorphism i.e., inverts the coproduct, in the sense that

$$\triangle \circ S = (S \otimes S) \circ \triangle^{op}\,, \tag{A.3}$$

Here $\triangle^{op}(u) = u_{(2)} \otimes u_{(1)}$ for $\triangle(u) = u_{(1)} \otimes u_{(2)}$. Furthermore instead of the Hopf algebra postulate $S(u_{(1)})\,u_{(2)} = 1_v \epsilon(u)$ the antipode of a WHA satisfies

$$S(u_{(1)})\,u_{(2)} \otimes u_{(3)} = (1_v \otimes u) \triangle(1_v) \, \big(= 1_{v\,(1)} \otimes u\,1_{v\,(2)} \big)\,, \tag{A.4}$$

The relations (A.3), (A.4) are checked using both unitarity relations (3.5), as well as (3.13),(3.14); the choice of the coefficient in (3.21) is essential.

One turns $\mathcal{A}$ into a *quasitriangular WHA* by defining an $\mathcal{R}$ - matrix, i.e. an element $\mathcal{R} \in \triangle^{op}(1_v)\ \mathcal{A} \otimes \mathcal{A}\ \triangle(1_v)$, which intertwines the two coproducts,

$$\triangle^{op}(u)\,\mathcal{R} = \mathcal{R}\,\triangle(u)\,, \tag{A.5}$$

subject to the constraints,

$$(\triangle \otimes Id)\,\mathcal{R} = \mathcal{R}_{13}\,\mathcal{R}_{23}\,, \qquad (Id \otimes \triangle)\,\mathcal{R} = \mathcal{R}_{13}\,\mathcal{R}_{12}\,. \tag{A.6}$$

Namely

$$\mathcal{R} = \sum_{\substack{i,j,p \\ a,a',b,c,c',d \\ \alpha,\alpha',\gamma,\gamma',\beta,\beta',t}} {}^{(1)}F_{bp} \begin{bmatrix} i & j \\ c' & a' \end{bmatrix}_{\alpha'\,\gamma'}^{\beta'\,t} w^{i,j,p}_{a,a';c,c';\beta,\beta'}\ {}^{(1)}F^*_{dp} \begin{bmatrix} j & i \\ c & a \end{bmatrix}_{\alpha\,\gamma}^{\beta\,t} e^{(ba')\,(cd)}_{j\,,\gamma'\,,\alpha} \otimes e^{(c'b)\,(da)}_{i\,,\alpha'\,,\gamma} \tag{A.7}$$

Here $w^{i,j,p}_{a,a';c,c';\beta,\beta'}$ is a unitary matrix and to make contact with the CFT under consideration we choose

$$w^{i,j,p}_{a,a';c,c';\beta,\beta'} = \delta_{\beta\beta'}\,\delta_{aa'}\,\delta_{cc'}\ e^{-i\pi\triangle^p_{ij}}\,, \tag{A.8}$$

so that the coefficient in (A.7) reproduces the braiding matrix $\hat{B}(+)$ in (4.14). Similarly one defines $\mathcal{R}^* \in \triangle(1_v)\ \mathcal{A} \otimes \mathcal{A}\ \triangle^{op}(1_v)$, corresponding to $\hat{B}(-)$; the inversion relation (4.10) is equivalent to $\mathcal{R}^*\,\mathcal{R} = \triangle(1_v)\,,\ \mathcal{R}\,\mathcal{R}^* = \triangle^{op}(1_v)$. The relations (A.6) are equivalent to

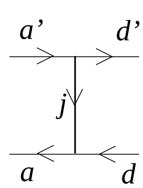
the fusing-braiding relation (4.17) and its counterpart discussed in section 4. Denoting by P the permutation operator in $V^i \otimes V^j$, the definition (A.7) with the choice (A.8) implies

$$P\,\mathcal{R}\,\frac{1}{\sqrt{P_b}}\,e^{i,\alpha'}_{cb}\otimes_h e^{j,\gamma'}_{ba} = \sum_{\substack{d,\\ \alpha,\gamma}} \hat{B}^*_{bd}\begin{bmatrix} i & j\\ c & a\end{bmatrix}^{\alpha\,\gamma}_{\alpha'\,\gamma'}(-)\,\frac{1}{\sqrt{P_d}}\,e^{j,\alpha}_{cd}\otimes_h e^{i,\gamma}_{da}\,. \tag{A.9}$$

The horizontal product in $\mathcal{A}$ depicted on Fig. 6 reads more explicitly

$$e^{(cb)(c'b')}_{i\,,\alpha\,,\alpha'}\otimes_h e^{(da)(d'a')}_{j\,,\gamma\,,\gamma'} = \delta_{bd}\,\delta_{b'd'}\sum_{\substack{p\\ \beta,\beta',t}} g^{p;b,b'}_{ij}\;{}^{(1)}F_{bp}\begin{bmatrix} i & j\\ c & a\end{bmatrix}^{\beta\,t}_{\alpha\,\gamma}\;{}^{(1)}F^*_{b'p}\begin{bmatrix} i & j\\ c' & a'\end{bmatrix}^{\beta'\,t}_{\alpha'\,\gamma'}\; e^{(ca)(c'a')}_{p\,,\beta\,,\beta'}\,. \tag{A.10}$$

The dual algebra $\hat{\mathcal{A}}$ of $\mathcal{A}$ is the space of linear functionals on $\mathcal{A}$. It is a matrix algebra $\hat{\mathcal{A}} = \oplus_{x\in\widetilde{\mathcal{V}}} Mat_{\widetilde{m}_x}$ with matrix unit basis $\{E_x^{(a'a;\eta)(d'd;\zeta)}\,,\, x\in\widetilde{\mathcal{V}}\}$, depicted by double triangles, or blocks, with an intermediate index x see Figs. 2, 9. The indices $(a,a';\eta)$, $a,a'\in\mathcal{V}$, $\eta = 1,2,\dots \tilde{n}_{ax}{}^{a'}$ label the states in a linear vector space $\hat{V}_x$ of dimension $\dim(\hat{V}_x) = \sum_{a,a'}\tilde{n}_{ax}{}^{a'} = \tilde{m}_x$. They are depicted in Fig. 1 as triangles with two white and 1 black vertices. The vertical and horizontal products are exchanged in the dual algebra, i.e., the horizontal product is the matrix product in $\hat{\mathcal{A}}$ and the vertical product for the basis elements E_x is given by a dual analogue of (A.10), with the convention that the second element appears above the first. In this product the rôle of the multiplicities N_j and n_j is taken over by $\widetilde{N}_x$ and $\tilde{n}_x$, with the relation (1.3) serving now as a consistency relation replacing (1.1). The dual $3j$- and $6j$-symbols ${}^{(1)}\tilde{F}$ and $\tilde{F}$, the last two of the tetrahedra in Fig. 1, satisfy unitarity relations analogous to (3.5) and two more pentagon relations parallel to (3.9) and (3.7) respectively. The matrix $\tilde{F}$ dual to the fusing matrix F has all indices of type x, while ${}^{(1)}\tilde{F}_{bz}\left[\begin{smallmatrix} y & x\\ c & a\end{smallmatrix}\right]$ is a matrix with 3+3 indices of type $a,b,c\in\mathcal{V}$ and $x,y,z\in\widetilde{\mathcal{V}}$. All the steps of section 2 can be repeated, in particular we can choose a gauge fixing for ${}^{(1)}\tilde{F}$ analogous to (3.11), using that $\tilde{d}_x\,P_a = \sum_b \tilde{n}_{ax}{}^b\,P_b$.



$$= c^{da'}_j \sum_x {}^{(2)}F_{da'}\begin{bmatrix} j & d'\\ a & x\end{bmatrix}$$

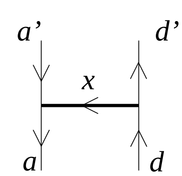


Fig. 9: Relating the two bases

The finite dimensional algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ can be identified, looking at $\{e_j^{\alpha,\alpha'}\}$ and $\{E_x^{\eta,\zeta}\}$ as providing different bases, see Fig. 9. This introduces a new "fusing" matrix ${}^{(2)}F$, given, up to a constant, by the numerical value of the linear functional $E_x^{\eta,\zeta}(e_j^{\alpha,\alpha'}) \in \mathbb{C}$. ${}^{(2)}F$ is the third tetrahedron on Fig. 1, supported by two black and two white vertices, and two types of triangles of multiplicities n_j and $\tilde{n}_x$. More explicitly we have

$$E_x^{\eta\,\zeta}(e_j^{\alpha\,\alpha'}) = E_{x;\eta\,,\zeta}^{(a'a)(d'd)}\Big(e_{j;\alpha\,\alpha'}^{(cb)(c'b')}\Big) = \delta_{ac}\,\delta_{bd}\delta_{a'c'}\,\delta_{b'd'}\,c_j^{da'}\,\tilde{c}_x^{da'}\;{}^{(2)}F_{b\,c'}\begin{bmatrix} j & b' \\ c & x \end{bmatrix}_{\alpha\,\zeta}^{\eta\,\alpha'} \tag{A.11}$$

with

$$c_j^{bc'}\;\tilde{c}_x^{bc'} = \frac{\tilde{d}_x\,d_j}{P_b\,P_{c'}}\,(\frac{S_{11}}{\psi_1^1})^2\,. \tag{A.12}$$

The equality (7.30) ensures that the number of elements on both sides of Fig. 9 for fixed a, a', d, d' and varying j and x is the same, so the linear transformation ${}^{(2)}F$ is invertible, the inverse denoted ${}^{(2)}\tilde{F}^*$, in the sense of the relations

$$\begin{gathered}\sum_{x,\eta,\zeta} c_j^{bc'}\;\tilde{c}_x^{bc'}\;{}^{(2)}F_{b\,c'}\begin{bmatrix} j & b' \\ c & x \end{bmatrix}_{\alpha\,\zeta}^{\eta\,\beta}\;{}^{(2)}\tilde{F}^*_{b\,c'}\begin{bmatrix} j' & b' \\ c & x \end{bmatrix}_{\alpha'\,\zeta}^{\eta\,\beta'} = \delta_{jj'}\,\delta_{\alpha\,\alpha'}\,\delta_{\beta\beta'} \\ \sum_{j,\alpha,\beta} c_j^{bc'}\;\tilde{c}_x^{bc'}\;{}^{(2)}F_{b\,c'}\begin{bmatrix} j & b' \\ c & x' \end{bmatrix}_{\alpha\,\zeta}^{\eta\,\beta}\;{}^{(2)}\tilde{F}^*_{b\,c'}\begin{bmatrix} j & b' \\ c & x \end{bmatrix}_{\alpha\,\zeta'}^{\eta'\,\beta} = \delta_{xx'}\,\delta_{\zeta\,\zeta'}\,\delta_{\eta\,\eta'}\,.\end{gathered} \tag{A.13}$$

We shall require that ${}^{(2)}F$ and ${}^{(2)}\tilde{F}$ are trivial for $x = 1$ and $j = 1$, analogously to (3.4). This is consistent with the inverse relations (A.13), inserting (A.12) and using that

$$\sum_x \tilde{n}_{ax}{}^b\;\tilde{d}_x = (\frac{\psi_1^1}{S_{11}})^2\,P_a\,P_b = \sum_j n_{ja}{}^b\;d_j\,. \tag{A.14}$$

We recall that the ratio of constants $c_j^{bc'}$ appears in the normalisation of the horizontal product (A.10), and similarly a ratio of the constants $\tilde{c}_x^{bc'}$ determines the constant in the vertical product of the dual basis elements. Inserting the relation in Fig. 9 in both sides of the horizontal product (A.10) and using furthermore that the horizontal product acts trivially on the dual basis by a formula analogous to (3.22), one gets the pentagon relation [1]

$$\begin{gathered}\sum_{b',\alpha',\gamma',\zeta} {}^{(1)}F_{b'p}\begin{bmatrix} i & j \\ a' & c' \end{bmatrix}_{\gamma'\,\alpha'}^{\beta'\,t}\;{}^{(2)}F_{ba'}\begin{bmatrix} i & b' \\ a & x \end{bmatrix}_{\gamma\,\zeta}^{\eta\,\gamma'}\;{}^{(2)}F_{cb'}\begin{bmatrix} j & c' \\ b & x \end{bmatrix}_{\alpha\,\eta'}^{\zeta\,\alpha'} \\ = \sum_\beta {}^{(1)}F_{bp}\begin{bmatrix} i & j \\ a & c \end{bmatrix}_{\gamma\,\alpha}^{\beta\,t}\;{}^{(2)}F_{ca'}\begin{bmatrix} p & c' \\ a & x \end{bmatrix}_{\beta\,\eta'}^{\eta\,\beta'}\,.\end{gathered} \tag{A.15}$$

In terms of the functional values (A.11) the identity (A.15) reads [1]

$$\sum_{\zeta} E_x^{\eta\,\zeta}(e_i^{\gamma\,\alpha})\; E_{x'}^{\zeta\,\eta'}(e_j^{\gamma'\,\alpha'}) = \delta_{xx'}\; E_x^{\eta\,\eta'}(e_i^{\gamma\,\alpha} \otimes_h\, e_j^{\gamma'\,\alpha'})\,. \tag{A.16}$$

Similarly starting from the vertical product analogue of (A.10) for the dual basis we obtain the dual analogue of (A.15), with ${}^{(1)}F$ and ${}^{(2)}F$ replaced by ${}^{(1)}\tilde{F}$ and ${}^{(2)}\tilde{F}$. The relation in Fig. 9 allows to define a sesquilinear form in the algebra determined by the pairing (A.11) on $\mathcal{A}\otimes\hat{\mathcal{A}}$, s.t. $\langle E_x\,,\, E_{x'}\rangle = \delta_{xx'}\,\tilde{c}_x$. Assuming furthermore that $\langle e_j\,,\, e_{j'}\rangle = \delta_{jj'}\,c_j$ leads to the identification ${}^{(2)}F = {}^{(2)}\tilde{F}$. Then the above two dual pentagon identities are equivalent to the identities relating, via the pairing, the coproduct in each of the two algebras to the product in its dual [4],

$$\begin{aligned} \langle E_x E_y\,,\, e_p\rangle &= \langle E_x\otimes E_y\,,\, \triangle(e_p)\rangle \\ \langle E_z\,,\, e_i e_j\rangle &= \langle \triangle(E_z)\,,\, e_i\otimes e_j\rangle\,, \end{aligned} \tag{A.17}$$

where the products in the l.h.s. stand for the algebra multiplications in $\hat{\mathcal{A}}$ and $\mathcal{A}$ (i.e., the horizontal and vertical products respectively). [10] The coproduct and the horizontal product are related via the scalar product in $\mathcal{A}$ defined above

$$\langle e_i \otimes_h\, e_j\,,\, e_p\rangle = \langle e_i\otimes e_j\,,\, \triangle(e_p)\rangle\,. \tag{A.18}$$

We shall furthermore assume the analogues of the symmetry relations (3.13), (3.14) (compatible with the form of ${}^{(2)}F$ and with the relation $\tilde{n}_{ax}{}^{b} = \tilde{n}_{bx^*}{}^{a}$)

$${}^{(2)}F_{b\,c'}\begin{bmatrix} j & b' \\ c & x\end{bmatrix} = \sqrt{\frac{P_b\,P_{c'}}{P_{b'}\,P_c}}\;{}^{(2)}F^{*}_{c\,b'}\begin{bmatrix} j^* & c' \\ b & x\end{bmatrix} = {}^{(2)}F_{c'\,b}\begin{bmatrix} j^* & c \\ b' & x^*\end{bmatrix}\,, \tag{A.19}$$

Inserting the first equality of (A.19) in the relation obtained from (A.15) for $p=1$, one obtains using (3.11)

$$\sum_{b',\zeta,\gamma'} {}^{(2)}F^{*}_{a\,b'}\begin{bmatrix} j & a' \\ b & x\end{bmatrix}^{\zeta\,\gamma'}_{\gamma\,\eta}\;{}^{(2)}F_{c\,b'}\begin{bmatrix} j & a' \\ b & x\end{bmatrix}^{\zeta\,\eta'}_{\alpha\,\zeta'} = \delta_{ac}\,\delta_{\alpha\gamma}\,\delta_{\eta\eta'}\;. \tag{A.20}$$

[10] The above identification ${}^{(2)}F = {}^{(2)}\tilde{F}$ appears in [4] (up to different notation) as a solution in the diagonal cases, where the r.h.s. of (A.12) simplifies to a ratio of q–dimensions. In general the matrix defining the pairing on $\mathcal{A}\otimes\hat{\mathcal{A}}$ and its inverse are left unrelated and the equalities (A.17) lead to dual (with respect to the $3j$-symbols) pentagons in both of which only the inverse matrix enters. We are indebted to Gabriella Böhm and Kornél Szlachányi for a clarifying e-mail correspondence on this point.

Using (A.19) one also checks that the conjugation operation $^+$ computed directly from (A.11) coincides with $E^+(e) = \overline{E(S^{-1}(e)^+)}$. To make contact with the basis for the dual triangles exploited in [4] one has to introduce $\phi_x^{(cc')(bb')} := \sqrt{\frac{P_b\, P_{c'}}{P_{b'}\, P_c}}\, E_x^{(cc')(bb')} = S(E_{x^*}^{(b'b)(c'c)})$, so that $\langle \phi^+, e\rangle = \overline{\langle \phi, S(e)^+\rangle}$.

The identity (A.15) and its dual complete the set of pentagon type relations called "the Big Pentagon" in [4]. In the diagonal case $Z_{j\bar{j}} = \delta_{j\bar{j}}$ where all multiplicities $N_i\,, n_i\,, \tilde{N}_x$ and $\tilde{n}_x$ coincide with the Verlinde one, one can identify ${}^{(1)}F = F = {}^{(2)}F = {}^{(1)}\tilde{F} = \tilde{F}$ since all pentagon relations involved coincide with (3.9) and the unitarity relations (3.5) and their dual counterparts, as well as (A.13), reduce to the unitarity of F. The next simple cases are the permutation modular invariants $Z_{j\bar{j}} = Z^{\rm diag}_{j\zeta(\bar{j})}$, where ζ is an automorphism of the fusion rules. For any of these cases $\widetilde{\mathcal{V}}$ is identified with $\mathcal{I}$, $\widetilde{N} = N\,, \tilde{n} = n$, and accordingly $\tilde{F} = F\,,\ {}^{(1)}\tilde{F} = {}^{(1)}F$. We notice that in these cases the pentagon identity (A.16) looks like the fusing-braiding identity (4.16) and this suggests that given ${}^{(1)}F$, and hence by (4.14), given $\hat{B}$, the latter matrix may provide, up to some constant, a solution for ${}^{(2)}F$. In the simplest example of the $D_{\rm odd}$ $sl(2)$ series the matrices ${}^{(1)}F$ were computed in [6].

Defining the dual counterparts of (3.22)

$$\hat{E}_x = \sum_{c\,,b\,,\eta} \frac{1}{\hat{c}_x^{bc}}\ E^{(cb)(cb)}_{x\,,\eta\,,\eta}\,, \tag{A.21}$$

and using the analogues of (A.10) and (3.5) with ${}^{(1)}F$ replaced by ${}^{(1)}\tilde{F}$ one obtains the algebra (7.16) with the multiplication identified with the vertical product

$$\hat{E}_x \otimes_v\ \hat{E}_y = \sum_z\ \tilde{N}_{xy}{}^z\, \hat{E}_z\,. \tag{A.22}$$

The identity in $\hat{\mathcal{A}}$ is given by $\mathbf{1}_h = \sum_{x\,,c\,,b\,,\eta}\ E^{(cb)(cb)}_{x\,,\eta\,,\eta}$ and (A.11), (A.13) ensure that $\mathbf{1}_h$ coincides with the counit $\frac{1}{|\mathcal{I}|}\,\epsilon$. [11] The dual algebra $\hat{\mathcal{A}}$ cannot be turned in general into a quasitriangular WHA.

---

[11] The factor $1/P_bP_{c'}$ in (A.12), dictated by the requirement of consistency of the full set of pentagon and inversion equations, can be assigned entirely to one of the constants c_j or $\hat{c}_x$. Then one of the formulae (3.22) or (A.21) gives elements in the center of the corresponding algebra, the so called "minimal central projections". However there seems to exist no consistent renormalisation of the two products and of the relation on Fig. 9 making central the basis elements of both algebras (3.23) and (A.22).

The relations (A.22), (7.5), (1.5), imply that the "chiral generators"

$$p_j^+ = \sum_x \tilde{V}_{j1;\,1}{}^x \, \hat{E}_x \,, \qquad p_j^- = \sum_x \tilde{V}_{1j^*;\,1}{}^x \, \hat{E}_x \,, \tag{A.23}$$

satisfy the Verlinde algebra

$$p_i^{\pm} \otimes_v p_j^{\pm} = \sum_k N_{ij}{}^k \, p_k^{\pm} \,, \tag{A.24}$$

while

$$p_i^+ \otimes_v p_j^- = \sum_z \mathbf{T}_{ij}^z \, \hat{E}_z \,. \tag{A.25}$$

Since $\frac{1}{|\tilde{\mathcal{V}}|}\,\tilde{\varepsilon}(\hat{E}_z) = \delta_{z1}$, applying $\frac{1}{|\tilde{\mathcal{V}}|}\,\tilde{\varepsilon}$ to (A.25) reproduces in the r.h.s. the modular invariant matrix $Z_{ij} = \mathbf{T}_{ij}^1$ [1]; in [11] the analogous relation reads $\langle \alpha_i^+, \alpha_j^- \rangle = Z_{ij}$.

Appendix B. The $sl(2)$ theories

In this appendix, we illustrate the construction of Ocneanu graphs and of the associated matrix algebras on the $\widehat{sl}(2)$ theories and modular invariants of ADE type.

The cases of A_n and $D_{2\ell+1}$ have been covered by the discussion in section 7.5: the A cases are diagonal theories and the $D_{2\ell+1}$ case is obtained from the diagonal case $A_{4\ell-1}$ with the same Coxeter number $h = 4\ell$ by the $\mathbb{Z}_2$ automorphism $\zeta(j) = h - j$ of the fusion rules. The $D_{2\ell}$, E_6 and E_8 cases have also been implicitly covered there. But we shall collect here additional data on them and present the E_7 case which does not follow from the previous formulae. Throughout this appendix, we follow the notations of [3] on the vertices and on the eigenvectors of ordinary Dynkin diagrams.

For the $D_{2\ell}$ theories, in which condition (7.43) is satisfied, formula (7.41) applies with $b = 1$ and $\kappa = 0, 1$. Diagonalising the matrices $\tilde{V}_1{}^x$ as explained in section 7.5, and with a little extra insight to find the appropriate combinations of eigenvectors Ψ with exponent $(J) = (j, \bar{j}) = (2\ell-1, 2\ell-1)$ of multiplicity $(Z_{j\bar{j}})^2 = 4$, we find that

$$\{\Psi_x^{(J;\alpha,\beta)}\} = \Bigg(\underbrace{\begin{matrix} \frac{\psi_a^j}{\sqrt{2}} & \frac{\psi_a^j}{\sqrt{2}} & \frac{\psi_a^{h-j}}{\sqrt{2}} & \frac{\psi_a^{h-j}}{\sqrt{2}} \\ \frac{\psi_a^j}{\sqrt{2}} & -\frac{\psi_a^j}{\sqrt{2}} & -\frac{\psi_a^{h-j}}{\sqrt{2}} & \frac{\psi_a^{h-j}}{\sqrt{2}} \end{matrix}}_{j=1,3,\cdots,2\ell-3} \begin{matrix} \psi_a^{(2\ell-1,+)} & \psi_a^{(2\ell-1,-)} & 0 & 0 \\ 0 & 0 & \psi_a^{(2\ell-1,+)} & \psi_a^{(2\ell-1,-)} \end{matrix}\Bigg) \tag{B.1}$$

which should be understood as follows: the exponent of ψ in the first four columns run over $j = 1, 3, \cdots, 2\ell-3$, ($h = 4\ell-2$ is the Coxeter number), and for each j the corresponding

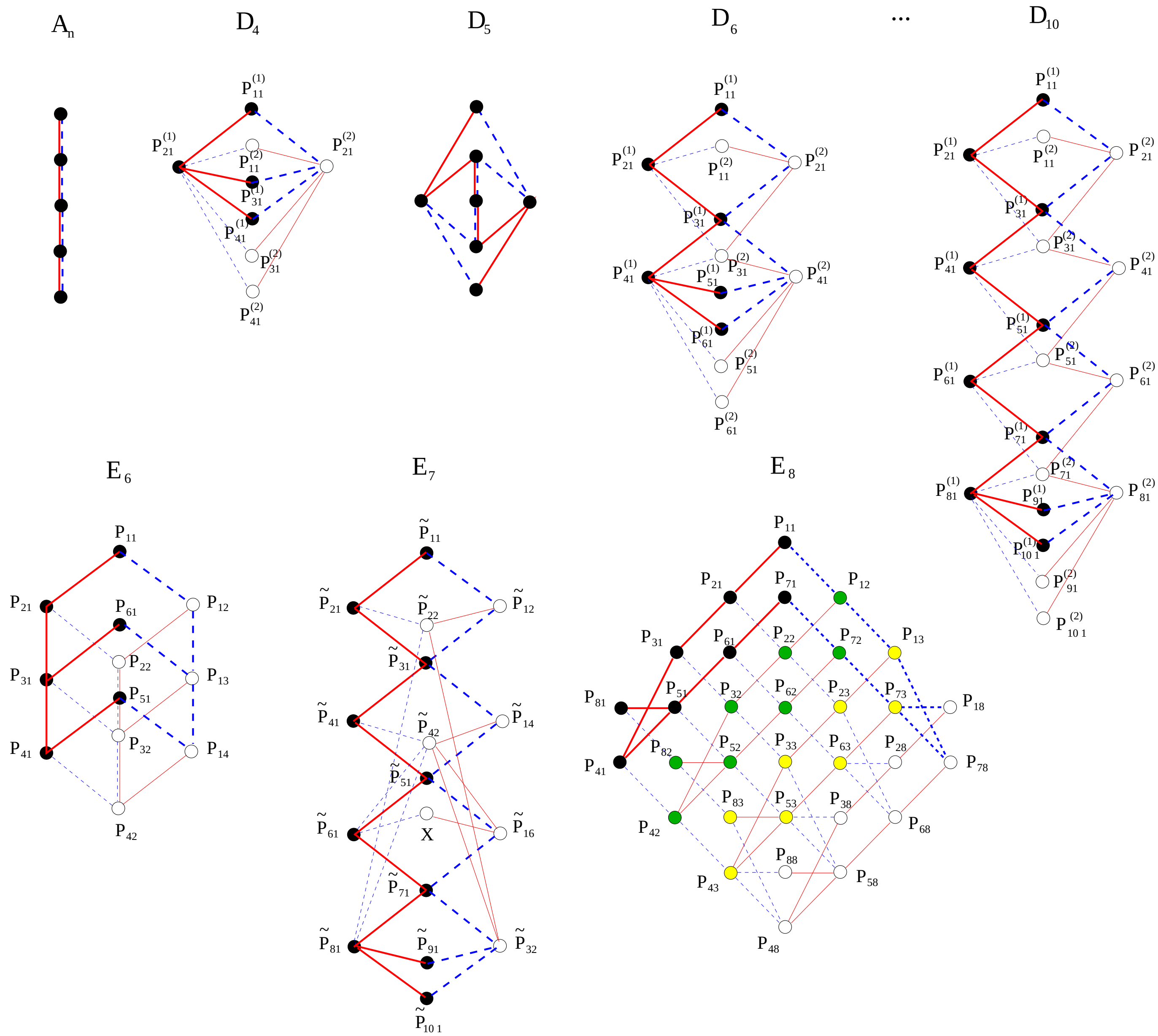


Fig. 10: The Ocneanu graphs of ADE type: each vertex x is assigned its matrix $\tilde{V}_1^x$, written as a P or $\tilde{P}$ matrix as in (7.41) or in (B.9). Edges of $\tilde{V}_{21}$, resp $\tilde{V}_{12}$ are shown in red full lines, resp blue broken ones, and the vertices of the different "cosets" for the action of $\tilde{V}_{21}$ are depicted in different colours.

value of the pair $(J) = (j, \bar{j})$ is successively (j, j), $(j, h-j), (h-j, j)$, $(h-j, h-j)$. In the last four columns, the exponent of Ψ is $(\frac{h}{2}, \frac{h}{2}; \alpha, \beta)$ with successively $(\alpha, \beta) = (1,1), (2,2,), (1,2), (2,1)$. The row index x is of the form $x = (a, \kappa)$, as in sections 7.5, and the first line of (B.1) refers to $\kappa = 0$, the second to $\kappa = 1$.

It is then easy to compute the various sets of matrices discussed in section 7. One

finds that

$$\begin{aligned}\tilde{V}_{ij} &= \begin{pmatrix} n_i\, n_j & 0 \\ 0 & n_i\, n_j \end{pmatrix} \text{ if } j \text{ is odd} \\ &= \begin{pmatrix} 0 & n_i\, n_j \\ n_i\, n_j & 0 \end{pmatrix} \text{ if } j \text{ is even },\end{aligned} \tag{B.2}$$

$$\begin{aligned}\widetilde{N}_x &= \begin{pmatrix} \widehat{N}_a & 0 \\ 0 & \widehat{N}_{a^c} \end{pmatrix} \qquad \text{if } \kappa = 0 \\ &= \begin{pmatrix} 0 & \widehat{N}_a \\ \widehat{N}_{a^c} & 0 \end{pmatrix} \qquad \text{if } \kappa = 1\end{aligned} \tag{B.3}$$

and

$$\tilde{n}_x = \begin{cases} \widehat{N}_a & \text{if } \kappa = 0 \\ \widehat{N}_a C & \text{if } \kappa = 1 \end{cases} \tag{B.4}$$

where the index c in (B.3) denotes the $\mathbb{Z}_2$ involution of vertices of the (ordinary) $D_{2\ell}$ diagram which exchanges the two vertices of the fork and leaves the other invariant, and $C_{ab} = \delta_{ab^c}$. Using these data, one checks (7.31) and (7.32). Finally the matrices $\widetilde{M}$ restricted to the "physical" subset, i.e. those that do not involve $j = \bar{j} = 2\ell - 1$ with labels $\alpha \neq \beta$, are all non negative. For $j_1, \bar{j}_1$ etc $\neq \frac{h}{2} = 2\ell - 1$:

$$\begin{aligned}
&\widetilde{M}_{(j_1,\bar{j}_1)(j_2,\bar{j}_2)}{}^{(j_3,\bar{j}_3)} = \begin{cases} M_{j_1 j_2}{}^{j_3} & \text{if there is 0 or 2 pairs of } (j, h-j) \text{ among } (j_1,\bar{j}_1),(j_2,\bar{j}_2),(j_3,\bar{j}_3) \\ 0 & \text{otherwise} \end{cases} \\
&\widetilde{M}_{(j_1,\bar{j}_1)(j_2,\bar{j}_2)}{}^{(2\ell-1,2\ell-1;\,\alpha,\alpha)} = \frac{1}{\sqrt{2}} M_{j_1 j_2}{}^{(2\ell-1,\alpha)} \\
&\widetilde{M}_{(2\ell-1,2\ell-1,\alpha,\alpha),(2\ell-1,2\ell-1,\beta,\beta)}{}^{(J)} = M_{(2\ell-1,\alpha)(2\ell-1,\beta)}{}^{j} \\
&\widetilde{M}_{(2\ell-1,2\ell-1,\alpha,\alpha),(2\ell-1,2\ell-1,\beta,\beta)}{}^{(2\ell-1,2\ell-1,\gamma,\gamma)} = \sqrt{2} M_{(2\ell-1,\alpha)(2\ell-1,\beta)}{}^{(2\ell-1,\gamma)} \ ,
\end{aligned}$$

in terms of the "ordinary" Pasquier algebra structure constants $M_{j_1 j_2}{}^{j_3}$ for which explicit expressions can be found in the Appendix A of [17][12]. These expressions of $\widetilde{M}$ are in agreement with their connection with the relative structure constants (7.35).

We now turn to the three exceptional cases.

The case of E_6

In that case, it suffices to take $x = (a, b)$, $a = 1, \cdots, 6$, $b = 1, 2$ and the $\tilde{V}_1{}^x$ equal to the matrices $P_{ab} := P^{(1)}_{ab}$. According to what was stated above in eq (7.42), the two sets

[12] with unfortunately a misprint which we correct here: in the last line of (A.2), the $\frac{1}{2}$ should read $\frac{1}{\sqrt{2}}$.

$\{P_{a1}\}$ and $\{P_{a2}\}$ are separately closed upon the left action of N_2. Moreover, because of symmetries $P_{13} = P_{62}\,,\ P_{14} = P_{52}\ ,\ P_{16} = P_{61}\,,\ P_{15} = P_{51}\,,\ P_{32} = P_{23}\,,\ P_{42} = P_{24}$ the two sets may also be regarded as $\{P_{1a}\}$ and $\{P_{2a}\}$, $a = 1, \cdots 6$ and are separately closed upon right action of N_2. See figure 10 on which each vertex x of the graph $\widetilde{E_6}$ is assigned its matrix $\tilde{V}^x$.

Using (7.45), it is easy to compute the various sets of matrices discussed in section 7. One finds, in accordance with (7.46), that

$$
\begin{aligned}
\widetilde{N}_x &= \begin{pmatrix} \widehat{N}_a & 0 \\ 0 & \widehat{N}_a \end{pmatrix} \qquad \text{if } x = (a,1) \\
&= \begin{pmatrix} 0 & \widehat{N}_a \\ \widehat{N}_a & \widehat{N}_a \widehat{N}_6 \end{pmatrix} \qquad \text{if } x = (a,2)
\end{aligned}
\tag{B.6}
$$

and $\tilde{n}_x$ is given by the last equation (7.46). One also computes $m_j = 6, 10, 14, 18, 20, 20, 20\,,$ $18, 14, 10, 6$ for $j = 1, \cdots, 11$; $\widetilde{m}_x = 6, 10, 14, 10, 6, 8$ and $10, 20, 28, 20, 10, 14$ for $x = (a, b = 1)$ and $(a, 2)$ respectively, $a = 1, \cdots, 6$. Hence one checks (7.32): $\sum_j m_j = 156 = \sum_x \widetilde{m}_x$, and (7.31): $\sum_j m_j^2 = \sum_x \widetilde{m}_x^2 = 2512$. Finally the matrices $\widetilde{M}$ factorise into a product of ordinary Pasquier matrices

$$
\widetilde{M}_{(I)\,(J)}{}^{(K)} = M_{ij}{}^{k}\, M_{\bar{i}\bar{j}}{}^{\bar{k}}\ . \tag{B.7}
$$

The case of E_8

In that case there are $8 \times 8/2 = 32$ matrices that may be taken either as the four sets $\{P_{a1}\}$, $\{P_{a2}\}$, $\{P_{a3}\}$ and $\{P_{a8}\}$, $a = 1, 2, \cdots, 8$, or using once again their symmetries, as $\{P_{1a}\}$, $\{P_{2a}\}$, $\{P_{3a}\}$ and $\{P_{8a}\}$, $a = 1, 2, \cdots, 8$. See Fig 10.

One then computes

$$\begin{aligned}
\widetilde{N}_x &= \begin{pmatrix} \widehat{N}_a & 0 & 0 & 0 \\ 0 & \widehat{N}_a & 0 & 0 \\ 0 & 0 & \widehat{N}_a & 0 \\ 0 & 0 & 0 & \widehat{N}_a \end{pmatrix} \qquad \text{if } x=(a,1) \\
&= \begin{pmatrix} 0 & \widehat{N}_a & 0 & 0 \\ \widehat{N}_a & 0 & \widehat{N}_a & 0 \\ 0 & \widehat{N}_a & 0 & \widehat{N}_7\widehat{N}_a \\ 0 & 0 & \widehat{N}_7\widehat{N}_a & 0 \end{pmatrix} \qquad \text{if } x=(a,2) \\
&= \begin{pmatrix} 0 & 0 & \widehat{N}_a & 0 \\ 0 & \widehat{N}_a & 0 & \widehat{N}_7\widehat{N}_a \\ \widehat{N}_a & 0 & (\widehat{N}_1+\widehat{N}_7)\widehat{N}_a & 0 \\ 0 & \widehat{N}_7\widehat{N}_a & 0 & \widehat{N}_7\widehat{N}_a \end{pmatrix} \qquad \text{if } x=(a,3) \\
&= \begin{pmatrix} 0 & 0 & 0 & \widehat{N}_a \\ 0 & 0 & \widehat{N}_7\widehat{N}_a & 0 \\ 0 & \widehat{N}_7\widehat{N}_a & 0 & \widehat{N}_7\widehat{N}_a \\ \widehat{N}_a & 0 & \widehat{N}_7\widehat{N}_a & 0 \end{pmatrix} \qquad \text{if } x=(a,8)
\end{aligned} \tag{B.8}$$

and $\tilde{n}_x$ as in (7.46). Also $m_j = m_{30-j} = 8,14,20,26,32,38,44,48,52,56,60,62,64,64,64$ for $j = 1,\cdots,15$; $\widetilde{m}_x = (8,14,20,26,32,22,12,16)$, $(14,28,40,52,64,44,22,32)$, $(20,40,60,\ 78,96,64,32,48)$ and $(16,32,48,64,78,52,26,40)$ for $x = (a,b=1)$, $(a,2)$, $(a,3)$ and $(a,8)$, respectively, $a = 1,\cdots,8$. Hence one checks $\sum_j m_j = 1240 = \sum_x \widetilde{m}_x$, $\sum_j m_j^2 = \sum_x \widetilde{m}_x^2 = 63136$. Finally the matrices $\widetilde{M}$ factorise again into a product of ordinary Pasquier matrices, like in (B.7).

The case of E_7

This case is known to be related to the D_{10} case. The $P^{(1)}_{ab}$ matrices of D_{10} were defined in (7.41) by

$$(P^{(1)}_{ab})_{ij} = \sum_{c\in T=\{1,3,5,7,9,10\}} n_{ia}{}^c n_{jb}{}^c \ ,$$

with n the solutions of (1.1) pertaining to D_{10}. Using the same matrices, let us now define the $\tilde{P}$ matrices (twisted version of the P's) by

$$(\tilde{P}_{ab})_{ij} = \sum_{c\in\{1,3,5,7,9,10\}} n_{ia}{}^c n_{jb}{}^{\zeta(c)} \ , \tag{B.9}$$

with ζ the usual involution acting on the vertices of T

$$\{1,3,5,7,9,10\} \mapsto \{1,9,5,7,3,10\}.$$

As in section 7 (equation (7.42)), we have the property that upon left (resp right) multiplication by N_2, $N_2.\tilde{P}_{ab} = \sum_{a'} n_{2a}{}^{a'}(\tilde{P}_{a'b})$, (resp $\tilde{P}_{ab}.N_2 = \sum_{b'}(\tilde{P}_{ab'})n_{2b'}{}^{b}$.) Recall that here n_2 is the adjacency matrix of D_{10}. This relation explains the D_{10} pattern of the two chiral parts of the Ocneanu graph $\widetilde{E_7}$ on Fig. 10: the red full (resp blue broken) thick line represents left (resp right) fusion by N_2 and connects the matrices $\tilde{P}_{a1}$, (resp $\tilde{P}_{1a}$), $a = 1, \cdots 10$.

These matrices have to be supplemented by others to produce the full set of matrices $\mathbf{T}^x$ and the second part (the "coset" [1]) of the graph $\widetilde{E_7}$. Using the symmetries $(\tilde{P}_{ab})^T = \tilde{P}_{ba}$ etc of the matrices $\tilde{P}$, we find that starting with matrix $\tilde{P}_{12}$, left multiplication by N_2 produces the chain of matrices forming the coset

$$\begin{array}{c} \qquad\qquad\qquad\quad\;\; \tilde{P}_{16} \\ \qquad\qquad\qquad\quad\;\; \uparrow \\ \tilde{P}_{12} \to \tilde{P}_{22} \to \tilde{P}_{32} = \tilde{P}_{18} \to \tilde{P}_{42} = \tilde{P}_{24} \to \tilde{P}_{14} \to X := \tilde{P}_{26} - \tilde{P}_{24} \ , \end{array}$$

where the splitting of $\tilde{P}_{52}$ into the sum $\tilde{P}_{14} + \tilde{P}_{16}$ has formed the triple point of the E_7 diagram. The matrix X itself may be expressed as a bilinear form in the matrices n (relative to D_{10})

$$X = \sum_{a,b=2,4,6,8} (\widehat{N}_7 - \widehat{N}_9)_a{}^b \, n_{i1}{}^a n_{j1}{}^b \tag{B.10}$$

in such a way that the pairs (a, b) that are summed over are

$$(a, b) \in \{(4, 4), (6, 6), (8, 8), (2, 6), (6, 2), (4, 8), (8, 4), (6, 8), (8, 6)\} \ .$$

One finds, following downward first the D_{10} subgraph, and then the E_7 coset

$$\begin{aligned} &\widetilde{N}_x = \begin{pmatrix} \widehat{N}_x & 0 \\ 0 & n_x^{(E_7)} \end{pmatrix}, \ x = 1, 8; \qquad \widetilde{N}_9 = \begin{pmatrix} \widehat{N}_9 & 0 \\ 0 & n_9^{(E_7)} - n_3^{(E_7)} \end{pmatrix}; \\ &\widetilde{N}_{10} = \begin{pmatrix} \widehat{N}_{10} & 0 \\ 0 & n_3^{(E_7)} \end{pmatrix}, \qquad \widetilde{N}_x = \begin{pmatrix} 0 & \check{n}_{x-10} \\ \check{n}^T_{x-10} & 0 \end{pmatrix}, \ x = 11, \cdots, 17 \ . \end{aligned} \tag{B.11}$$

where $n^{(E_7)}$ denote the n-matrices of E_7, and $\widehat{N}$ are relative to D_{10}; $\check{n}_b$, $b = 1, \cdots, 7$, are seven 10×7 rectangular matrices intertwining the D_{10} and E_7 adjacency matrices (see [3], sec. 3.3, for a formula),

$$\begin{aligned} &\tilde{n}_x = n_i^{(E_7)}, \ x = 1, 8; \ \tilde{n}_9 = n_3^{(E_7)}; \tilde{n}_{10} = n_9^{(E_7)} - n_3^{(E_7)}; \ \tilde{n}_{11} = n_2^{(E_7)}; \tilde{n}_{12} = n_1^{(E_7)} + n_3^{(E_7)}; \\ &\tilde{n}_{13} = n_8^{(E_7)}; \tilde{n}_{14} = n_3^{(E_7)} + n_5^{(E_7)}; \tilde{n}_{15} = n_8^{(E_7)}; \tilde{n}_{16} = n_3^{(E_7)} + n_5^{(E_7)}; \tilde{n}_{17} = n_7^{(E_7)} - n_3^{(E_7)} \ . \end{aligned}$$

One also computes $m_i = m_{18-i} = 7, 12, 17, 22, 27, 30, 33, 34, 35$ for $1 \leq i \leq 9$, and $\widetilde{m}_x = 7, 12, 17, 22, 27, 30, 33, 34, 17, 18;\ 12, 24, 34, 44, 30, 16, 22$, so that $\sum_i m_i = \sum \widetilde{m}_x = 399$, $\sum_i m_i^2 = \sum_x \widetilde{m}_x^2 = 10905$. Finally, the $\widetilde{M}$ matrices may also be computed, and yield non negative numbers $(0, \frac{1}{4}, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{3}{4}, 1, \sqrt{2}, 2)$, which match what was computed on the relative structure constants d^2.

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